Network Embedding as Matrix Factorization: Unifying DeepWalk, LINE, PTE, and node2vec

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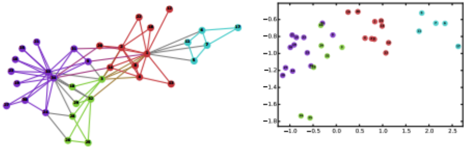
Motivation and Problem Formulation

Problem Formulation

Give a network $G = (V, E)$, aim to learn a function $f : V \to \mathbb{R}^p$ to capture neighborhood similarity and community membership.

Applications:

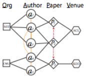
- link prediction
- community detection
- label classification



(a) Input: Karate Graph (b) Output: Representation

Figure 1: A toy example (Figure from DeepWalk).

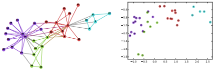
History of Network Embedding



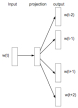
2017 metapath2vec [Dong et al.]

2016 node2vec [Grover & Leskovec]

2015 LINE & PTE [Tang et al.]



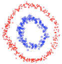
2014 DeepWalk [Perozzi et al.]



word2vec (skip-gram) [Mikolov et al.] 2013

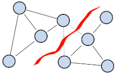
2009 SocDim [Tang & Liu]

Spectral Clustering v.s. Kernel k-means [Dhillon et al.] 2005



Spectral Clustering [Ng et al.] 2002

Image Segmentation [Shi & Malik] 2000



1996 A large body of literature
[Pothen et al.] [Simon] [Bolla],
[Hagen & Kahng] [Hendrickson & Leland]
[Van Driessche & Roose], [Barnard et al.]
[Spielman & Teng], [Guattery & Miller]

1973 Fiedler Vector [Fiedler]
Spectral Partitioning [Donath, Hoffman]

Contents

Notations

Consider an undirected weighted graph $G = (V, E)$, where $|V| = n$ and $|E| = m$.

- Adjacency matrix $\boldsymbol{A} \in \mathbb{R}_{+}^{n \times n}$:

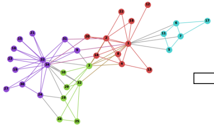
$$\boldsymbol{A}_{i,j} = \begin{cases} a_{i,j} > 0 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}.$$

- Degree matrix $\boldsymbol{D} = \operatorname{diag}(d_1, \cdots, d_n)$, where d_i is the generalized degree of vertex i.
- Volume of the graph G: $\operatorname{vol}(G) = \sum_i \sum_j \boldsymbol{A}_{i,j}$.

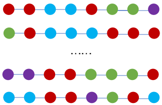
Assumption

$G = (V, E)$ is connected, undirected, and not bipartite, which makes $P(w) = \frac{d_w}{\operatorname{vol}(G)}$ a unique stationary distribution.

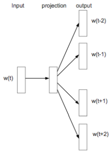
DeepWalk — Roadmap



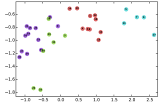
Input
G=(V,E)



Random
Walk



Skip-gram



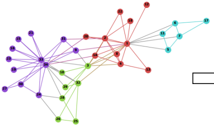
Output:
Node
Embedding

DeepWalk — a Two-step Algorithm

Algorithm 1: DeepWalk

for $n = 1, 2, \ldots, N$ **do**
    Pick w_1^n according to a probability distribution $P(w_1)$;
    Generate a vertex sequence $(w_1^n, \cdots, w_L^n)$ of length L by a random walk on network G;
    for $j = 1, 2, \ldots, L - T$ **do**
        for $r = 1, \ldots, T$ **do**
            Add vertex-context pair (w_j^n, w_{j+r}^n) to multiset $\mathcal{D}$;
            Add vertex-context pair (w_{j+r}^n, w_j^n) to multiset $\mathcal{D}$;
Run SGNS on $\mathcal{D}$ with b negative samples.

DeepWalk — Roadmap



Input
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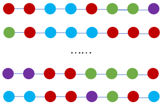

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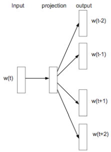

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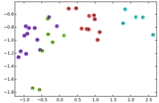


Output:
Node
Embedding

Levy & Goldberg (NIPS 14)

$$\log\left(\frac{\#(w,c)\,|\mathcal{D}|}{b\#(w)\#(c)}\right)$$

b Number of negative samples

$\#(w,c)$ Co-occurrence of w and c

$|\mathcal{D}|$ Total number of word-context pairs

$\#(w)$ Occurrence of word w

$\#(c)$ Occurrence of context c

Skip-gram with Negative Sampling (SGNS)

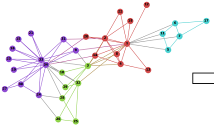
- SGNS maintains a multiset $\mathcal{D}$ which counts the occurrence of each word-context pair (w, c).
- Objective:

$$\mathcal{L} = \sum_w \sum_c \left(\#(w,c) \log g\left(\boldsymbol{x}_w^\top \boldsymbol{y}_c\right) + \frac{b\#(w)\#(c)}{|\mathcal{D}|} \log g\left(-\boldsymbol{x}_w^\top \boldsymbol{y}_c\right) \right),$$

  where $\boldsymbol{x}_w, \boldsymbol{y}_c \in \mathbb{R}^d$, g is the sigmoid function, and b is the number of negative samples for SGNS.
- For sufficiently large dimensionality d, equivalent to factorizing PMI matrix (Levy & Goldberg, NIPS'14):

$$\log\left(\frac{\#(w,c)\,|\mathcal{D}|}{b\#(w)\#(c)}\right).$$

DeepWalk — Roadmap



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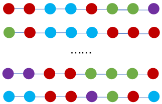

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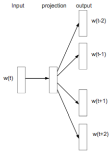
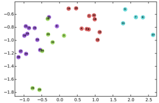


Skip-gram



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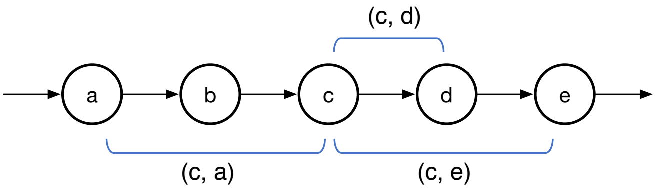
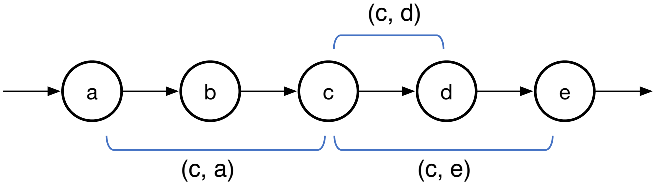


Question

Suppose the multiset $\mathcal{D}$ is constructed based on random walk on graph, can we interpret $\log\left(\frac{\#(w,c)|\mathcal{D}|}{b\#(w)\#(c)}\right)$ with graph theory terminologies?

DeepWalk



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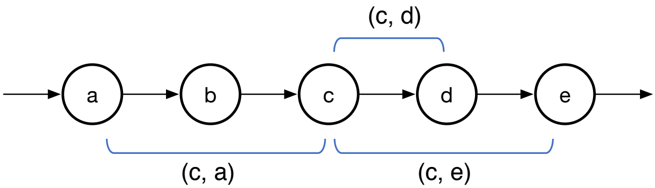
Challange

We mix so many things together, i.e., direction and distance.

DeepWalk



Question

Suppose the multiset $\mathcal{D}$ is constructed based on random walk on graph, can we interpret $\log\left(\frac{\#(w,c)|\mathcal{D}|}{b\#(w)\#(c)}\right)$ with graph theory terminologies?

Challange

We mix so many things together, i.e., direction and distance.

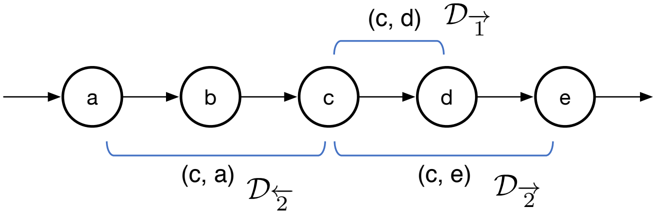
Solution

Let's distinguish them!

DeepWalk

Partition the multiset $\mathcal{D}$ into several sub-multisets according to the way in which vertex and its context appear in a random walk sequence. More formally, for $r = 1, \cdots, T$, we define

$$\mathcal{D}_{\overrightarrow{r}} = \left\{(w, c) : (w, c) \in \mathcal{D}, w = w_j^n, c = w_{j+r}^n\right\},$$
$$\mathcal{D}_{\overleftarrow{r}} = \left\{(w, c) : (w, c) \in \mathcal{D}, w = w_{j+r}^n, c = w_j^n\right\}.$$



DeepWalk as Implicit Matrix Factorization

Some observations

- Observation 1:

$$\log\left(\frac{\#(w,c)\,|\mathcal{D}|}{b\#(w)\cdot\#(c)}\right)=\log\left(\frac{\frac{\#(w,c)}{|\mathcal{D}|}}{b\frac{\#(w)}{|\mathcal{D}|}\frac{\#(c)}{|\mathcal{D}|}}\right)$$

- Observation 2:

$$\frac{\#(w,c)}{|\mathcal{D}|}=\frac{1}{2T}\sum_{r=1}^{T}\left(\frac{\#(w,c)_{\overrightarrow{r}}}{|\mathcal{D}_{\overrightarrow{r}}|}+\frac{\#(w,c)_{\overleftarrow{r}}}{|\mathcal{D}_{\overleftarrow{r}}|}\right).$$

Sufficient to characterize $\frac{\#(w,c)_{\overrightarrow{r}}}{|\mathcal{D}_{\overrightarrow{r}}|}$ and $\frac{\#(w,c)_{\overleftarrow{r}}}{|\mathcal{D}_{\overleftarrow{r}}|}$.

DeepWalk — Theorems

Theorem

Denote $\boldsymbol{P} = \boldsymbol{D}^{-1}\boldsymbol{A}$*, when the length of random walk* $L \to \infty$*,*

$$\frac{\#(w,c)_{\overrightarrow{r}}}{|\mathcal{D}_{\overrightarrow{r}}|} \xrightarrow{p} \frac{d_w}{\mathrm{vol}(G)}\left(\boldsymbol{P}^r\right)_{w,c} \text{ and } \frac{\#(w,c)_{\overleftarrow{r}}}{|\mathcal{D}_{\overleftarrow{r}}|} \xrightarrow{p} \frac{d_c}{\mathrm{vol}(G)}\left(\boldsymbol{P}^r\right)_{c,w}.$$

Theorem

When the length of random walk $L \to \infty$*, we have*

$$\frac{\#(w,c)}{|\mathcal{D}|} \xrightarrow{p} \frac{1}{2T}\sum_{r=1}^{T}\left(\frac{d_w}{\mathrm{vol}(G)}\left(\boldsymbol{P}^r\right)_{w,c} + \frac{d_c}{\mathrm{vol}(G)}\left(\boldsymbol{P}^r\right)_{c,w}\right).$$

Theorem

For DeepWalk, when the length of random walk $L \to \infty$*,*

$$\frac{\#(w,c)\,|\mathcal{D}|}{\#(w)\cdot\#(c)} \xrightarrow{p} \frac{\mathrm{vol}(G)}{2T}\left(\frac{1}{d_c}\sum_{r=1}^{T}\left(\boldsymbol{P}^r\right)_{w,c} + \frac{1}{d_w}\sum_{r=1}^{T}\left(\boldsymbol{P}^r\right)_{c,w}\right).$$

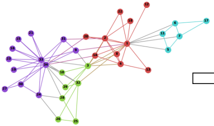
DeepWalk — Conclusion

Theorem

DeepWalk is asymptotically and implicitly factorizing

$$\log\left(\frac{\mathrm{vol}(G)}{b}\left(\frac{1}{T}\sum_{r=1}^{T}\left(\boldsymbol{D}^{-1}\boldsymbol{A}\right)^{r}\right)\boldsymbol{D}^{-1}\right).$$

DeepWalk — Roadmap



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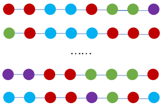

Random
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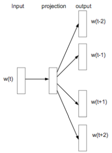
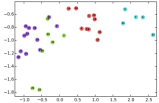


Skip-gram



Output:
Node
Embedding

Levy & Goldberg (NIPS 14)

$$\log\left(\frac{\#(w,c)\,|\mathcal{D}|}{b\#(w)\#(c)}\right)$$

$$\log\left(\frac{\mathrm{vol}(G)}{b}\left(\frac{1}{T}\sum_{r=1}^{T}(\boldsymbol{D}^{-1}\boldsymbol{A})^{r}\right)\boldsymbol{D}^{-1}\right)$$

$\boldsymbol{A}$ Adjacency matrix

$\boldsymbol{D}$ Degree matrix

$\mathrm{vol}(G)=\sum_i\sum_j \boldsymbol{A}_{i,j}$

b Number of negative samples

LINE

- Objective of LINE:

$$\mathcal{L} = \sum_{i=1}^{|V|} \sum_{j=1}^{|V|} \left(\boldsymbol{A}_{i,j} \log g\left(\boldsymbol{x}_i^\top \boldsymbol{y}_j\right) + \frac{b d_i d_j}{\operatorname{vol}(G)} \log g\left(-\boldsymbol{x}_i^\top \boldsymbol{y}_j\right) \right).$$

- Align it with the Objective of SGNS:

$$\mathcal{L} = \sum_{w} \sum_{c} \left(\#(w,c) \log g\left(\boldsymbol{x}_w^\top \boldsymbol{y}_c\right) + \frac{b\#(w)\#(c)}{|\mathcal{D}|} \log g\left(-\boldsymbol{x}_w^\top \boldsymbol{y}_c\right) \right).$$

- LINE is actually factorizing

$$\log\left(\frac{\operatorname{vol}(G)}{b} \boldsymbol{D}^{-1} \boldsymbol{A} \boldsymbol{D}^{-1}\right)$$

- Recall DeepWalk's matrix form:

$$\log\left(\frac{\operatorname{vol}(G)}{b} \left(\frac{1}{T} \sum_{r=1}^{T} \left(\boldsymbol{D}^{-1}\boldsymbol{A}\right)^r\right) \boldsymbol{D}^{-1}\right).$$

Observation LINE is a special case of DeepWalk ($T = 1$).

PTE

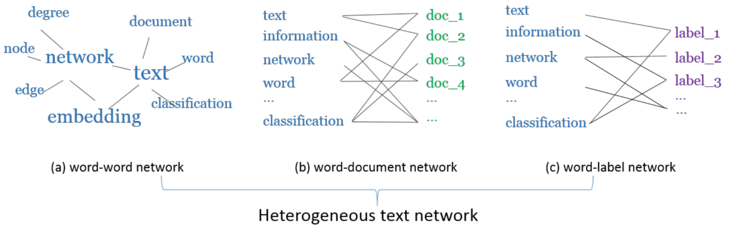


Figure 2: Heterogeneous Text Network.

- word-word network G_{ww}, $\boldsymbol{A}_{\text{ww}} \in \mathbb{R}^{\#\text{word}\times\#\text{word}}$.
- document-word network G_{dw}, $\boldsymbol{A}_{\text{dw}} \in \mathbb{R}^{\#\text{doc}\times\#\text{word}}$.
- label-word network G_{lw}, $\boldsymbol{A}_{\text{lw}} \in \mathbb{R}^{\#\text{label}\times\#\text{word}}$.

PTE as Implicit Matrix Factorization

$$\log \left(\begin{bmatrix} \alpha \operatorname{vol}(G_{\text{ww}}) (\boldsymbol{D}_{\text{row}}^{\text{ww}})^{-1} \boldsymbol{A}_{\text{ww}} (\boldsymbol{D}_{\text{col}}^{\text{ww}})^{-1} \\ \beta \operatorname{vol}(G_{\text{dw}}) (\boldsymbol{D}_{\text{row}}^{\text{dw}})^{-1} \boldsymbol{A}_{\text{dw}} (\boldsymbol{D}_{\text{col}}^{\text{dw}})^{-1} \\ \gamma \operatorname{vol}(G_{\text{lw}}) (\boldsymbol{D}_{\text{row}}^{\text{lw}})^{-1} \boldsymbol{A}_{\text{lw}} (\boldsymbol{D}_{\text{col}}^{\text{lw}})^{-1} \end{bmatrix} \right) - \log b,$$

- The matrix is of shape $(\#\text{word} + \#\text{doc} + \#\text{label}) \times \#\text{word}$.
- b is the number of negative samples in training.
- $\{\alpha, \beta, \gamma\}$ are hyper-parameters to balance the weights of the three networks. In PTE, $\{\alpha, \beta, \gamma\}$ satisfy

$$\alpha \operatorname{vol}(G_{\text{ww}}) = \beta \operatorname{vol}(G_{\text{dw}}) = \gamma \operatorname{vol}(G_{\text{lw}})$$

node2vec — 2nd Order Random Walk

$$\underline{\boldsymbol{T}}_{u,v,w} = \begin{cases} \frac{1}{p} & (u,v) \in E, (v,w) \in E, u = w; \\ 1 & (u,v) \in E, (v,w) \in E, u \neq w, (w,u) \in E; \\ \frac{1}{q} & (u,v) \in E, (v,w) \in E, u \neq w, (w,u) \notin E; \\ 0 & \text{otherwise.} \end{cases}$$

$$\underline{\boldsymbol{P}}_{u,v,w} = \operatorname{Prob}\left(w_{j+1} = u | w_j = v, w_{j-1} = w\right) = \frac{\underline{\boldsymbol{T}}_{u,v,w}}{\sum_u \underline{\boldsymbol{T}}_{u,v,w}}.$$

Stationary Distribution

$$\sum_w \underline{\boldsymbol{P}}_{u,v,w} \boldsymbol{X}_{v,w} = \boldsymbol{X}_{u,v}$$

Existence guaranteed by Perron-Frobenius theorem, but may not be unique.

node2vec as Implicit Matrix Factorization

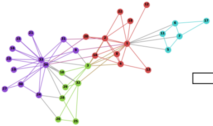
Theorem

node2vec is asymptotically and implicitly factorizing a matrix whose entry at w-th row, c-th column is

$$\log\left(\frac{\frac{1}{2T}\sum_{r=1}^{T}\left(\sum_u \boldsymbol{X}_{w,u}\underline{\boldsymbol{P}}^r_{c,w,u}+\sum_u \boldsymbol{X}_{c,u}\underline{\boldsymbol{P}}^r_{w,c,u}\right)}{b\left(\sum_u \boldsymbol{X}_{w,u}\right)\left(\sum_u \boldsymbol{X}_{c,u}\right)}\right)$$

Contents

Roadmap



Input
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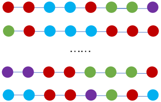
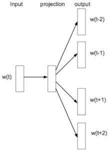
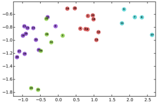


Random
Walk



Skip-gram



Output:
Node
Embedding

Levy & Goldberg (NIPS 14)

$$\log\left(\frac{\#(w,c)\,|\mathcal{D}|}{b\#(w)\#(c)}\right)$$

$$\log\left(\frac{\mathrm{vol}(G)}{b}\left(\frac{1}{T}\sum_{r=1}^{T}\left(\boldsymbol{D}^{-1}\boldsymbol{A}\right)^{r}\right)\boldsymbol{D}^{-1}\right)$$

Matrix
Factorization

NetMF

- Factorize the DeepWalk matrix:

$$\log\left(\frac{\mathrm{vol}(G)}{b}\left(\frac{1}{T}\sum_{r=1}^{T}\left(\boldsymbol{D}^{-1}\boldsymbol{A}\right)^{r}\right)\boldsymbol{D}^{-1}\right).$$

- For numerical reason, we use truncated logarithm — $\tilde{\log}(x) = \log\left(\max(1, x)\right)$

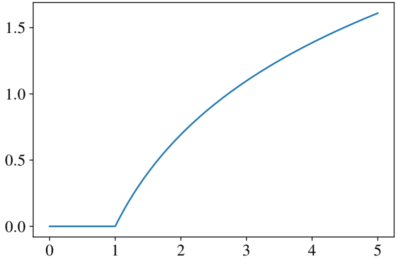


Figure 3: Truncated Logarithm

NetMF for a Small Window Size T

Algorithm 2: NetMF for a Small Window Size T

Compute $\boldsymbol{P}^1, \cdots, \boldsymbol{P}^T$;
Compute $\boldsymbol{M} = \frac{\mathrm{vol}(G)}{bT}\left(\sum_{r=1}^{T} \boldsymbol{P}^r\right)\boldsymbol{D}^{-1}$;
Compute $\boldsymbol{M}' = \max(\boldsymbol{M}, 1)$;
Rank-d approximation by SVD: $\log \boldsymbol{M}' = \boldsymbol{U}_d \boldsymbol{\Sigma}_d \boldsymbol{V}_d^\top$;
return $\boldsymbol{U}_d \sqrt{\boldsymbol{\Sigma}_d}$ as network embedding.

NetMF for a Large Window Size T — Observations

- We want to factorize

$$\tilde{\log}\left(\frac{\mathrm{vol}(G)}{b}\left(\frac{1}{T}\sum_{r=1}^{T}\left(\boldsymbol{D}^{-1}\boldsymbol{A}\right)^{r}\right)\boldsymbol{D}^{-1}\right).$$

- We know the property of normalized graph Laplacian

$$\boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}=\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{\top}$$

  where $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_n)$ and $\forall\lambda_i\in[-1,1]$.

$$\begin{aligned}\left(\frac{1}{T}\sum_{r=1}^{T}\left(\boldsymbol{D}^{-1}\boldsymbol{A}\right)^{r}\right)\boldsymbol{D}^{-1}&=\left(\boldsymbol{D}^{-1/2}\right)\left(\frac{1}{T}\sum_{r=1}^{T}\left(\boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}\right)^{r}\right)\left(\boldsymbol{D}^{-1/2}\right)\\&=\left(\boldsymbol{D}^{-1/2}\right)\left(\boldsymbol{U}\underbrace{\left(\frac{1}{T}\sum_{r=1}^{T}\boldsymbol{\Lambda}^{r}\right)}_{\text{A polynomial}}\boldsymbol{U}^{\top}\right)\left(\boldsymbol{D}^{-1/2}\right)\end{aligned}$$

NetMF for a Large Window Size T — Observations

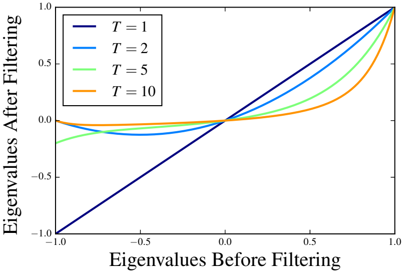


Figure 4: $f(\lambda) = \frac{1}{T} \sum_{r=1}^{T} \lambda^r$

Idea

This polynomial implicitly filters out negative eigenvalues and small positive eigenvalues, why not do it explicitly.

NetMF for a Large Window Size T — Algorithm

Algorithm 3: NetMF for a Large Window Size T

Eigen-decomposition $\boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2} \approx \boldsymbol{U}_h\boldsymbol{\Lambda}_h\boldsymbol{U}_h^\top$;

Approximate $\boldsymbol{M}$ with
$\hat{\boldsymbol{M}} = \frac{\mathrm{vol}(G)}{b}\boldsymbol{D}^{-1/2}\boldsymbol{U}_h\left(\frac{1}{T}\sum_{r=1}^{T}\boldsymbol{\Lambda}_h^r\right)\boldsymbol{U}_h^\top\boldsymbol{D}^{-1/2}$;

Compute $\hat{\boldsymbol{M}}' = \max(\hat{\boldsymbol{M}}, 1)$;

Rank-d approximation by SVD: $\log\hat{\boldsymbol{M}}' = \boldsymbol{U}_d\boldsymbol{\Sigma}_d\boldsymbol{V}_d^\top$;

return $\boldsymbol{U}_d\sqrt{\boldsymbol{\Sigma}_d}$ as network embedding.

Setup

Label Classification:

- BlogCatelog, PPI, Wikipedia, Flickr
- Logistic Regression
- NetMF $(T = 1)$ v.s. LINE
- NetMF $(T = 10)$ v.s. DeepWalk

Table 1: Statistics of Datasets.

Dataset	BlogCatalog	PPI	Wikipedia	Flickr
$\|V\|$	10,312	3,890	4,777	80,513
$\|E\|$	333,983	76,584	184,812	5,899,882
#Labels	39	50	40	195

Experimental Results

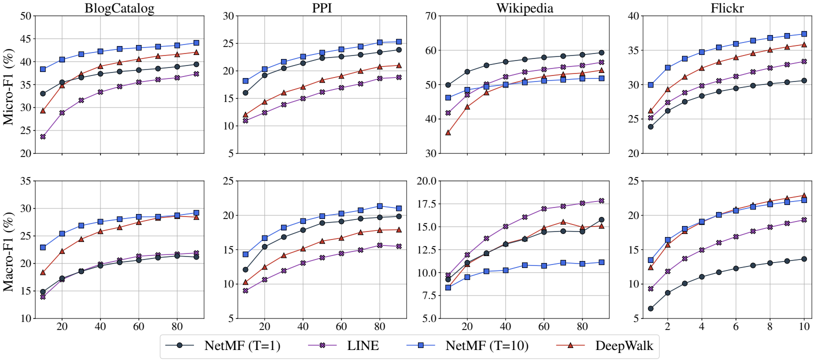


Figure 5: Predictive performance on varying the ratio of training data. The x-axis represents the ratio of labeled data (%), and the y-axis in the top and bottom rows denote the Micro-F1 and Macro-F1 scores respectively.

Conclusion

Table 2: The matrices that are implicitly approximated and factorized by DeepWalk, LINE, PTE, and node2vec.

Algorithm	Matrix
DeepWalk	$\log\left(\operatorname{vol}(G)\left(\frac{1}{T}\sum_{r=1}^{T}(\boldsymbol{D}^{-1}\boldsymbol{A})^r\right)\boldsymbol{D}^{-1}\right)-\log b$
LINE	$\log\left(\operatorname{vol}(G)\boldsymbol{D}^{-1}\boldsymbol{A}\boldsymbol{D}^{-1}\right)-\log b$
PTE	$\log\left(\begin{bmatrix}\alpha\operatorname{vol}(G_{\text{ww}})(\boldsymbol{D}_{\text{row}}^{\text{ww}})^{-1}\boldsymbol{A}_{\text{ww}}(\boldsymbol{D}_{\text{col}}^{\text{ww}})^{-1}\\ \beta\operatorname{vol}(G_{\text{dw}})(\boldsymbol{D}_{\text{row}}^{\text{dw}})^{-1}\boldsymbol{A}_{\text{dw}}(\boldsymbol{D}_{\text{col}}^{\text{dw}})^{-1}\\ \gamma\operatorname{vol}(G_{\text{lw}})(\boldsymbol{D}_{\text{row}}^{\text{lw}})^{-1}\boldsymbol{A}_{\text{lw}}(\boldsymbol{D}_{\text{col}}^{\text{lw}})^{-1}\end{bmatrix}\right)-\log b$
node2vec	$\log\left(\frac{\frac{1}{2T}\sum_{r=1}^{T}\left(\sum_u \boldsymbol{X}_{w,u}\underline{\boldsymbol{P}}_{c,w,u}^{r}+\sum_u \boldsymbol{X}_{c,u}\underline{\boldsymbol{P}}_{w,c,u}^{r}\right)}{\left(\sum_u \boldsymbol{X}_{w,u}\right)\left(\sum_u \boldsymbol{X}_{c,u}\right)}\right)-\log b$

Thanks.

Standing on the shoulders of giants
— Isaac Newton

Code available at `github.com/xptree/NetMF`
Q&A

DeepWalk — Sketched Proof

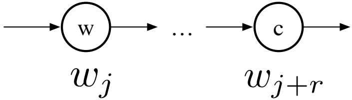
Theorem

Denote $\boldsymbol{P} = \boldsymbol{D}^{-1}\boldsymbol{A}$, when $L \to \infty$, we have

$$\frac{\#(w,c)_{\overrightarrow{r}}}{|\mathcal{D}_{\overrightarrow{r}}|} \xrightarrow{p} \frac{d_w}{\mathrm{vol}(G)} \left(\boldsymbol{P}^r\right)_{w,c} \text{ and } \frac{\#(w,c)_{\overleftarrow{r}}}{|\mathcal{D}_{\overleftarrow{r}}|} \xrightarrow{p} \frac{d_c}{\mathrm{vol}(G)} \left(\boldsymbol{P}^r\right)_{c,w}.$$

Proof.

Consider the special case when $N = 1$, thus we only have one vertex sequence $w_1, \cdots, w_L$ generated by random walk. Let Y_j $(j = 1, \cdots, L - T)$ be the indicator function for event that $w_j = w$ and $w_{j+r} = c$



Proof (Con't)

Observation

- $\mathbb{E}[Y_j] = \text{Prob}(w_j = w, w_{j+r} = c) \to \frac{d_w}{\text{vol}(G)} \left(\boldsymbol{P}^r\right)_{w,c}$.
- $\frac{\#(w,c)_{\overrightarrow{r}}}{|\mathcal{D}_{\overrightarrow{r}}|} = \frac{1}{L-T} \sum_{j=1}^{L-T} Y_j$.
- $\text{Cov}(Y_i, Y_j) \to 0$ as $|i-j| \to \infty$.

Lemma

(S.N. Bernstein Law of Large Numbers) Let $Y_1, Y_2 \cdots$ be a sequence of random variables with finite expectation $\mathbb{E}[Y_j]$ and variance $\text{Var}(Y_j) < K$, $j \geq 1$, and covariances are s.t. $\text{Cov}(Y_i, Y_j) \to 0$ as $|i-j| \to \infty$. Then the law of large numbers (LLN) holds.

$$\frac{\#(w,c)_{\overrightarrow{r}}}{|\mathcal{D}_{\overrightarrow{r}}|} = \frac{1}{L-T} \sum_{j=1}^{L-T} Y_j \xrightarrow{p} \frac{1}{L-T} \sum_{j=1}^{L-T} \mathbb{E}(Y_j) \to \frac{d_w}{\text{vol}(G)} \left(\boldsymbol{P}^r\right)_{w,c}$$

Time Complexity

- Eigen-Decomposition (Implicitly Restarted Lanczos Method) $O(mhI + nh^2I + h^3I)$.
- Reconstruction $O(n^2h)$
- Element-wise logarithm $O(n^2)$.
- SVD (a naive implementation with eigen-decomposition): $O(n^2dI + nd^2I + d^3I)$.

Future Work

- Comprehend high-order cases, e.g., node2vec.

$$\log \left(\frac{\frac{1}{2T}\sum_{r=1}^{T}\left(\sum_u \boldsymbol{X}_{w,u}\underline{\boldsymbol{P}}^r_{c,w,u} + \sum_u \boldsymbol{X}_{c,u}\underline{\boldsymbol{P}}^r_{w,c,u}\right)}{b\left(\sum_u \boldsymbol{X}_{w,u}\right)\left(\sum_u \boldsymbol{X}_{c,u}\right)} \right)$$

- Design scalable algorithm (e.g., using spectral sparsification of random-walk polynomials).

$$\log\left(\frac{\mathrm{vol}(G)}{b}\left(\frac{1}{T}\sum_{r=1}^{T}\left(\boldsymbol{D}^{-1}\boldsymbol{A}\right)^r\right)\boldsymbol{D}^{-1}\right).$$

- Connection with graph convolutional networks (Kipf & Welling, ICLR'17).