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Networks are prevalent in many areas and are often collected from multiple sources. However, due to the veracity characteristics, more often than not, networks are incomplete. Network alignment and network completion have become two fundamental cornerstones behind a wealth of high-impact graph mining applications. The state-of-the-art have been addressing these two tasks in parallel. That is, most of the existing network alignment methods have implicitly assumed that the topology of the input networks for alignment are perfectly known a priori, whereas the existing network completion methods admit either a single network (i.e., matrix completion) or multiple aligned networks (e.g., tensor completion). In this paper, we argue that network alignment and completion are inherently complementary with each other, and hence propose to jointly address them so that the two tasks can mutually benefit from each other. We formulate the problem from the optimization perspective, and propose an effective algorithm (INEAT) to solve it. The proposed method offers two distinctive advantages. First (Alignment accuracy), our method benefits from the higher-quality input networks while mitigates the effect of the incorrectly inferred links introduced by the completion task itself. Second (Alignment efficiency), thanks to the low-rank structure of the complete networks and the alignment matrix, the alignment process can be significantly accelerated. We perform extensive experiments which show that (1) the network completion can significantly improve the alignment accuracy, i.e., up to 30% over the baseline methods; (2) the network alignment can in turn help recover more missing edges than the baseline methods; and (3) our method achieves a good balance between the running time and the accuracy, and scales with a provable *linear* complexity in both time and space.

#### CCS Concepts: • Information systems → Data mining;

Additional Key Words and Phrases: Incomplete network alignment, network completion, low rank.

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#### **1 INTRODUCTION**

Networks are prevalent and naturally appear in many areas. More often than not, in the big data era, networks in many high-impact applications are collected from *multiple* sources (i.e., *variety*), such as social networks from different social platforms, protein-protein interaction (PPI) networks from

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multiple tissues, transaction networks from multiple financial institutes, etc. In order to integrate the considerable information associated with multiple networks, network alignment is of key importance to find the node correspondence across networks. For example, by aligning the same users in different transaction networks, the transaction patterns of users can be comprehended to enhance the financial fraud detection. However, real-world networks are often *incomplete* (i.e., *veracity*) due to, for instance, the difficulties in data collections. As such, network completion (e.g., to infer the missing links) has become another key task which benefits many graph mining applications by providing higher-quality networks if handled properly.

Although the multi-sourced and incomplete characteristics often *co-exist* in many real networks, the state-of-the-arts have been largely addressing network alignment and network completion problems *in parallel*. For example, most of the existing network alignment methods based on topological consistency have implicitly assumed that the topology of the input networks for alignment are perfectly known a priori [16, 42]. On the other hand, the existing network completion methods aim to infer the missing links in either a single network (e.g., by matrix completion [27]) or multiple networks that are aligned beforehand (e.g., by tensor completion [24]). How can we align two input incomplete networks when missing edges are unobserved in them?

A natural choice could be *completion-then-alignment*. That is, we first separately complete the missing edges in the input networks by some existing network completion methods, followed by the alignment across the resulting complete networks. However, there exist some fundamental limits of this strategy on the alignment performance. First (*Alignment accuracy*), the promise of this strategy lies in that by inferring the missing links of each input network, it would provide higher-quality input networks for the alignment task. However, the completion task itself might introduce noise (e.g., truly nonexistent edges), which might compromise, or even prevail the benefits of the correctly inferred missing links for the alignment task. Second (*Alignment efficiency*), the network alignment alone is already computationally costly. Most of the existing methods (even with approximation, such as [43]) have a time/space complexity at least  $O(n^2)$ , where *n* is the number of nodes of the input networks, mainly due to the computation/storage of the alignment matrix and the sparse matrix-matrix multiplication between the input network even denser by adding the missing edges. As a result, if we simply conduct the network alignment task on such densified networks, it might make the computation even more intensive.

To address these limitations, we hypothesize that network alignment and network completion can inherently complement each other due to the following reasons. First, (*H1*) alignment helps completion. Intuitively, when many nodes in one network share similar connectivity patterns with their corresponding aligned nodes (e.g., connecting to the similar sets of nodes) in another network, the knowledge about the existence or absence of links in one network could help inferring the missing links in another network via alignment if we can find such node correspondences across networks. Second, (*H2*) completion helps alignment. As introduced before, network completion could potentially improve the qualities of input networks, leading to the enhancement of the alignment accuracy. Moreover, network completion itself implicitly assumes a low-rank structure on the input networks, which, if harnessed appropriately, will actually accelerate the alignment process as we will show in the paper.

Armed with these hypotheses, we propose to *jointly* address network alignment and network completion problems so that the two tasks could mutually benefit from each other. To be specific, in order to leverage alignment for the completion task, we impose the low-rank structure on

<sup>&</sup>lt;sup>1</sup>Although the *empirical* runtime of some existing methods (e.g., *BigAlign* [16]) is *near-linear*, the big-O time complexity of these methods is still quadratic.

the underlying (true) network, which matches not only the observed links of the corresponding network, but also the *auxiliary observations* from the other network via the alignment matrix. Second, in order to leverage the network completion for the alignment, we recast the network alignment problem via the low-rank structures of the *complete* networks, which not only improves the alignment accuracy, but also speeds up the alignment process. We formulate them into a joint optimization problem and propose an effective algorithm to solve it.

The main contributions of the paper are summarized as:

- **Problem Definition.** To our best knowledge, we are the first to jointly address the network alignment and network completion tasks in an optimization framework.
- Algorithm and Analysis. We propose an effective algorithm (INEAT) based on the multiplicative update to solve the optimization. We also analyze its correctness, convergence and complexity. In particular, we prove that the low-rank structure of the complete networks guarantees a low-rank structure of the alignment matrix, which in turn reduces the time complexity of each iterative update to be *linear*. To our best knowledge, this is the first known network alignment algorithm with a provable linear time complexity.
- **Experiments.** We evaluate the effectiveness and efficiency of the proposed algorithm by extensive experiments. The experimental results demonstrate that (1) network alignment and network completion can indeed benefit from each other in terms of alignment accuracy and missing edges recovery rate, (2) our algorithm INEAT achieves a better alignment and completion quality, and meanwhile is faster than most of the baseline methods, and (3) our algorithm is only *linear* w.r.t. the number of nodes in the networks.

The rest of the paper is organized as follows. Section 2 defines the incomplete network alignment problem and provides some preliminaries of the paper. Section 3 presents the proposed optimization formulation of INEAT and Section 4 gives an effective optimization algorithm, followed by some analyses. Section 5 presents the experimental results. Related work and conclusion are given in Section 6 and Section 7.

## 2 PROBLEM DEFINITION

#### 2.1 Problem Definition

Table 1 summarizes the main symbols and notations used throughout the paper. We use bold uppercase letters for matrices (e.g., **A**), bold lowercase letters for vectors (e.g., **s**), and lowercase letters (e.g.,  $\alpha$ ) for scalars. We use **A**(*i*, *j*) to denote the entry at the intersection of the *i*-th row and *j*-th column of the matrix **A**. We denote the transpose of a matrix by a superscript *T* (e.g., **A**<sup>*T*</sup> is the transpose of **A**). The vectorization of a matrix (in the column order) is denoted by vec(·), and the result vector is denoted by the corresponding bold lowercase letter (e.g., **s** = vec(**S**)). Equivalently, the transformation of a vector to its corresponding matrix is denoted by a de-vectorization operator mat(·) (e.g., **S** = mat(**s**)). The trace of a matrix is denoted by Tr(·), and the diagonal matrix of a vector is denoted by diag(·).

Many real-world networks are incomplete with missing edges. Although some incompleteness scenarios may be possible (e.g., with the probabilities whether edges exist known a priori), in our paper, we only consider the network incompleteness where we only have the knowledge about the existence (i.e., a value of 1) or the absence (i.e., a value of 0) of certain entries (denoted by the set  $\Omega$ ) of its adjacency matrix. For the rest entries in the adjacency matrix, we do not know if the corresponding links exist or not, and hence are represented as the question mark ?. Figure 1 presents an illustrative example. All solid lines represent the observed existing edges. As we can see in Figure 1(a), the set of nodes (1, 2, 3, 4) in the first incomplete network have similar topology to the nodes (6', 7', 8', 9'), possibly leading to a wrong alignment result that these two sets of

Table 1.	Symbols and Notations
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Symbols	Definition
$\mathcal{G}_1,\mathcal{G}_2$	incomplete networks
$\mathbf{A}_1, \mathbf{A}_2$	two adjacency matrices of $\mathcal{G}_1$ and $\mathcal{G}_2$
$n_1, n_2$	# of nodes in $\mathcal{G}_1$ and $\mathcal{G}_2$
$m_1, m_2$	# of nodes in $\mathcal{G}_1$ and $\mathcal{G}_2$
S	an $n_2  imes n_1$ alignment matrix between $\mathcal{G}_2$ and $\mathcal{G}_1$
$P_{\Omega}(\cdot), P_{\bar{\Omega}}(\cdot)$	an operator to project only to observed (unobserved) entries
$\mathbf{U}_1, \mathbf{V}_1, \mathbf{U}_2, \mathbf{V}_2$	low rank factorizations of $A_1$ and $A_2$
$\mathbf{P}_{\Omega_1}, \mathbf{P}_{\Omega_2}$	projection matrix, all 1s at all observed entries
$1_1, 1_2$	1s vectors of length $n_1$ and $n_2$ respectively
$\lambda, \gamma, \beta$	parameters
$Tr[\cdot]$	trace operator
$diag(\cdot)$	diagonal matrix of a vector
$vec(\cdot), mat(\cdot)$	vectorization and de-vectorization operator
$rank(\cdot)$	the rank of a matrix
$eig(\cdot)$	eigenvalues of a matrix

nodes are aligned within each other. However, the complete networks in Figure 1(b) (by filling all the red lines) are identical, such as the cliques formed by nodes (1, 2, 3, 4) and (1', 2', 3', 4'). Thus, the set of nodes (1, 2, 3, 4) can be aligned to nodes (1', 2', 3', 4') respectively, so can the rest of nodes. On the other hand, by completing two networks separately, noisy edges might be incorrectly added (e.g., edge (4, 6)) and the true network structure would fail to be recovered. The incorrectly recovered networks may further mislead the alignment results. Therefore, how to align the incomplete networks while completing them is the key challenge this paper aims to address.

PROBLEM 1. INCOMPLETE NETWORK ALIGNMENT.

*Given:* (1) Incomplete adjacency matrices  $A_1$ ,  $A_2$  of two undirected networks  $G_1$ ,  $G_2$ , and (2-optional) a prior node similarity matrix H across networks.

**Output:** (1) the  $n_2 \times n_1$  alignment/similarity matrix S, where S(x, a) represents to what extent node-a in  $G_1$  is aligned with node-x in  $G_2$ , and (2) complete adjacency matrices  $A_1^*$ , and  $A_2^*$ .

#### 2.2 Preliminaries

A - Network Alignment. Most existing network alignment algorithms (such as *IsoRank* [34] and *FINAL* [43, 44]), explicitly or implicitly, are based on the *topology consistency* principle. Take *FINAL* as an example, the topology consistency principle can be stated as follows<sup>2</sup>. Given two pairs of nodes, say (1) node-*a* in  $\mathcal{G}_1$  and node-*x* in  $\mathcal{G}_2$  and (2) node-*b* in  $\mathcal{G}_1$  and node-*y* in  $\mathcal{G}_2$ , if nodes *a* and *b* are close neighbors and nodes *x* and *y* are also close neighbors, the *topology consistency* principle assumes the similarity between *a* and *x*, and that between their respective close neighbors *b* and *y* to be consistent, i.e., small  $[\hat{S}(a, x) - \hat{S}(b, y)]^2 A_1(a, b) A_2(x, y)$ , where  $\hat{S}$  is the similarity matrix. Mathematically, this naturally leads to the following optimization problem.

$$\min_{\mathbf{\alpha}} \alpha \hat{\mathbf{s}}^T (\mathbf{D} - \mathbf{A}_1 \otimes \mathbf{A}_2) \hat{\mathbf{s}} + (1 - \alpha) \| \mathbf{D}^{\frac{1}{2}} (\hat{\mathbf{s}} - \mathbf{h}) \|_F^2$$
(1)

where  $\hat{s}$ , h are the vectorization of the similarity matrix  $\hat{S}$  and the prior similarity matrix H respectively.  $D = D_1 \otimes D_2$  and  $D_1, D_2$  are the diagonal degree matrix of  $A_1, A_2$  respectively. Note that

 $<sup>^{2}</sup>$ In [43], the authors generalize the topology consistency principle to further accommodate the additional node/edge attribute information, which is outside the scope of this paper



Fig. 1. An illustrative example. Figure 1(a) shows the input incomplete networks and Figure 1(b) shows part of the alignment across two complete networks.

instead of using  $\hat{S}$  to infer the alignment as in [43], we use the scaled similarity matrix S as the 'soft' alignment matrix throughout our paper where S is the matrix form of  $s = D\hat{s}$  (i.e.,  $S = mat(D\hat{s})$ ). In other words, the entries in the alignment matrix S measure to what extent the two corresponding nodes are aligned together. Besides, the second regularization term in Eq. (1) is to avoid trivial solutions, such as a zero matrix  $\hat{S}$ .

In order to solve the network alignment problem in Eq. (1), we can either use an iterative algorithm with a time complexity of O(nm) and a space complexity  $O(n^2)$ , or resort to its closed-form solution whose time complexity could be as high as  $O(n^6)$  where we assume that the two networks have a comparable number of edges and nodes, i.e.,  $O(m) = O(m_1) = O(m_2)$  and  $O(n) = O(n_1) = O(n_2)$ . In [43], the authors proposed to approximate the closed-form solution via eigenvalue decomposition. But it is still *quadratic* in both time and space.

*B* - *Network Completion.* As mentioned earlier, incomplete networks might have many unobserved missing edges, which could significantly change the true network structure and hence mislead the topology-based network alignment. One straightforward way to address this issue is by using matrix completion. Most of the existing matrix completion methods are centered around minimizing the nuclear norm of the matrix [32]. However, since real-world networks are usually very large, it is very costly to directly minimize the nuclear norm of the adjacency matrices. In [33], the authors show that the nuclear norm  $\|\mathbf{A}_1\|_* = \min_{\mathbf{U}_1, \mathbf{V}_1} \frac{1}{2}(\|\mathbf{U}_1\|_F^2 + \|\mathbf{V}_1\|_F^2)$  where  $\mathbf{A}_1 = \mathbf{U}_1\mathbf{V}_1^T$ , which allows the factorization-based completion methods. To be specific, we can recover the complete networks by minimizing the following objective function:

$$J_{1}(\mathbf{U}_{1}, \mathbf{V}_{1}, \mathbf{U}_{2}, \mathbf{V}_{2}) = \underbrace{\frac{1}{2} \|P_{\Omega_{1}}(\mathbf{A}_{1} - \mathbf{U}_{1}\mathbf{V}_{1}^{T})\|_{F}^{2} + \frac{\lambda}{2}(\|\mathbf{U}_{1}\|_{F}^{2} + \|\mathbf{V}_{1}\|_{F}^{2})}_{\text{network completion on } \mathbf{A}_{1}} + \underbrace{\frac{1}{2} \|P_{\Omega_{2}}(\mathbf{A}_{2} - \mathbf{U}_{2}\mathbf{V}_{2}^{T})\|_{F}^{2} + \frac{\lambda}{2}(\|\mathbf{U}_{2}\|_{F}^{2} + \|\mathbf{V}_{2}\|_{F}^{2})}_{\text{network completion on } \mathbf{A}_{2}}$$
(2)

where the operator  $P_{\Omega_1}$  projects the value to the observed set  $\Omega_1$  of  $\mathbf{A}_1$ , e.g.,  $P_{\Omega_1}((\mathbf{U}_1\mathbf{V}_1^T)(i, j)) = (\mathbf{U}_1\mathbf{V}_1^T)(i, j)$  for any  $(i, j) \in \Omega_1$ , otherwise 0; and operator  $P_{\Omega_2}$  is defined similarly.

## **3 PROPOSED OPTIMIZATION FORMULATION**

In this section, we present the proposed optimization formulation to solve Problem 1. First, we present how to formulate the alignment task in the form of two complete networks. A key contribution here is that we prove that the low-rank structure of the complete networks guarantees a

low-rank structure of the alignment matrix. Then we present how to leverage the alignment matrix to infer missing edges across networks, followed by the overall optimization formulation.

#### 3.1 Network Completion Helps Network Alignment

By performing the network completion on both incomplete networks, the structure of the underlying networks could be recovered so that we can perform the alignment task across higher-quality networks. We use the factorization-based network completion (i.e., Eq (2)) and denote these two complete networks by  $\mathbf{A}_1^* = \mathbf{U}_1 \mathbf{V}_1^T$  and  $\mathbf{A}_2^* = \mathbf{U}_2 \mathbf{V}_2^T$ , where  $\mathbf{U}_i$  and  $\mathbf{V}_i$  (i = 1, 2) are the factorization matrices of rank-r. We adopt Eq. (1) to perform the network alignment task. Note that in general, we cannot guarantee the recovered adjacency matrices ( $\mathbf{A}_1^*$  and  $\mathbf{A}_2^*$ ) to be symmetric because  $\mathbf{V}_1$  ( $\mathbf{V}_2$ ) may not be identical to  $\mathbf{U}_1$  ( $\mathbf{U}_2$ ). This leads to a slightly different objective function from Eq. (1) to align directed networks. Specifically, based on the *topology consistency* (i.e., small [ $\hat{\mathbf{S}}(a, x) - \hat{\mathbf{S}}(b, y)$ ]<sup>2</sup> $\mathbf{A}_1(a, b)\mathbf{A}_2(x, y)$  in two directed networks), the optimization problem is formulated as follows.

$$\min_{\boldsymbol{\Lambda}} \alpha \hat{\boldsymbol{s}}^T (\hat{\boldsymbol{D}} - \boldsymbol{A}_1^* \otimes \boldsymbol{A}_2^*) \hat{\boldsymbol{s}} + (1 - \alpha) \| \hat{\boldsymbol{D}}^{\frac{1}{2}} (\hat{\boldsymbol{s}} - \boldsymbol{h}) \|_F^2$$
(3)

where  $\hat{\mathbf{D}} = \frac{\mathbf{D}_1 \otimes \mathbf{D}_2 + \hat{\mathbf{D}}_1 \otimes \hat{\mathbf{D}}_2}{2}$ ,  $\mathbf{D}_1 = \text{diag}(\mathbf{U}_1 \mathbf{V}_1^T \mathbf{1}_1)$  and  $\hat{\mathbf{D}}_1 = \text{diag}(\mathbf{1}_1^T \mathbf{U}_1 \mathbf{V}_1^T)$  are the outdegree matrix and indegree matrix of  $\mathbf{A}_1^*$ , respectively.  $\mathbf{D}_2$  and  $\hat{\mathbf{D}}_2$  are defined in a similar way.

However, directly solving the above problem requires at least  $O(n^2)$  time complexity, even with approximation. To address this issue, we give the following lemma, which states the alignment matrix **S** under the *topology consistency* (i.e., Eq. (3)) intrinsically consists of a *low-rank* structure, thanks to the low-rank structure of two complete adjacency matrices.

LEMMA 1. Low-Rank Structure of the Alignment Matrix S. Let  $\hat{s}$  be the solution of Eq. (3) where  $A_1^* = U_1 V_1^T$  and  $A_2^* = U_2 V_2^T$  are two complete rank-r adjacency matrices. Let the alignment matrix S be the scaled similarity matrix  $S = mat(\hat{D}\hat{s})$  and H be the prior similarity matrix, then if  $\alpha < 0.5$ , the alignment matrix can be expressed as  $S = \alpha U_2 M U_1 + (1 - \alpha) H$  where M is an  $r_2 \times r_1$  matrix and  $r_1$ ,  $r_2$  are the ranks of  $A_1^*$  and  $A_2^*$ , respectively.

PROOF. Followed by Eq. (3), the closed-form solution of similarity matrix  $\hat{S}$  can be computed by using Woodbury matrix identity [31] as below

$$\hat{\mathbf{s}} = (1-\alpha)\hat{\mathbf{D}}^{-1}\mathbf{h} + \alpha(1-\alpha)\hat{\mathbf{D}}^{-1}\mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{V}^{T}\hat{\mathbf{D}}^{-1}\mathbf{h}$$
(4)

where  $\mathbf{U} = \mathbf{U}_1 \otimes \mathbf{U}_2$ ,  $\mathbf{V} = \mathbf{V}_1 \otimes \mathbf{V}_2$ ,  $\mathbf{\Lambda} = \mathbf{I} - \alpha \mathbf{V}^T \hat{\mathbf{D}}^{-1} \mathbf{U}$ .

First, we rewrite  $\Lambda^{-1}$  as follows. Since for any two matrices X, Y, the eigenvalues of their product satisfies eig(XY) = eig(YX) [31], we obtain

$$|\operatorname{eig}(\alpha \mathbf{V}^T \hat{\mathbf{D}}^{-1} \mathbf{U})| = |\operatorname{eig}(\alpha \mathbf{U} \mathbf{V}^T \hat{\mathbf{D}}^{-1})|$$
  

$$\leq |\operatorname{eig}(2\alpha \mathbf{U} \mathbf{V}^T (\mathbf{D}_1 \otimes \mathbf{D}_2)^{-1})|$$
  

$$= 2\alpha |\operatorname{eig}((\mathbf{U}_1 \mathbf{V}_1^T \mathbf{D}_1^{-1}) \otimes (\mathbf{U}_2 \mathbf{V}_2^T \mathbf{D}_2^{-1}))|$$

Here, the term  $\mathbf{U}_1\mathbf{V}_1^T\mathbf{D}_1^{-1}$  represents a weighted directed network whose adjacency matrix has eigenvalues within (-1, 1), so as the term  $\mathbf{U}_2\mathbf{V}_2^T\mathbf{D}_2^{-1}$ . Thus, if  $\alpha < 0.5$ , according to the spectrum property of Kronecker product, we have

$$2\alpha |\operatorname{eig}((\mathbf{U}_1\mathbf{V}_1^T\mathbf{D}_1^{-1}) \otimes (\mathbf{U}_2\mathbf{V}_2^T\mathbf{D}_2^{-1}))| < 1$$

Then, we can use Neumann expansion on  $\Lambda^{-1}$  as

$$\mathbf{\Lambda}^{-1} = \sum_{k=0}^{\infty} (2\alpha)^k \left[ \mathbf{V}^T (2\hat{\mathbf{D}})^{-1} \mathbf{U} \right]^k$$
(5)

Next, we rewrite  $(2\hat{\mathbf{D}})^{-1}$  as follows. Denote  $\bar{\mathbf{D}}_1 = \mathbf{D}_1 + \hat{\mathbf{D}}_1$  and  $\bar{\mathbf{D}}_2 = \mathbf{D}_2 + \hat{\mathbf{D}}_2$ , we have  $(2\hat{\mathbf{D}})^{-1} = (\mathbf{D}_1 \otimes \mathbf{D}_2 + \hat{\mathbf{D}}_1 \otimes \hat{\mathbf{D}}_2)^{-1} = \{(\bar{\mathbf{D}}_1 \otimes \bar{\mathbf{D}}_2) | \mathbf{I} - (\bar{\mathbf{D}}_1^{-1} \otimes \bar{\mathbf{D}}_2^{-1}) (\mathbf{D}_1 \otimes \hat{\mathbf{D}}_2 + \hat{\mathbf{D}}_1 \otimes \mathbf{D}_2) ] \}^{-1}$ 

$$= [\mathbf{I} - (\bar{\mathbf{D}}_{1}^{-1} \otimes \bar{\mathbf{D}}_{2}^{-1})(\mathbf{D}_{1} \otimes \hat{\mathbf{D}}_{2} + \hat{\mathbf{D}}_{1} \otimes \mathbf{D}_{2})]^{-1}(\bar{\mathbf{D}}_{1}^{-1} \otimes \bar{\mathbf{D}}_{2}^{-1})$$

$$= \sum_{j=0}^{\infty} [(\bar{\mathbf{D}}_{1}^{-1}\mathbf{D}_{1}) \otimes (\bar{\mathbf{D}}_{2}^{-1}\hat{\mathbf{D}}_{2}) + (\bar{\mathbf{D}}_{1}^{-1}\hat{\mathbf{D}}_{1}) \otimes (\bar{\mathbf{D}}_{2}^{-1}\mathbf{D}_{2})]^{j}(\bar{\mathbf{D}}_{1}^{-1} \otimes \bar{\mathbf{D}}_{2}^{-1})$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{j} {j \choose i} [(\bar{\mathbf{D}}_{1}^{-1}\mathbf{D}_{1})^{i}(\bar{\mathbf{D}}_{1}^{-1}\hat{\mathbf{D}}_{1})^{j-i}\bar{\mathbf{D}}_{1}^{-1}] \otimes [(\bar{\mathbf{D}}_{2}^{-1}\hat{\mathbf{D}}_{2})^{i}(\bar{\mathbf{D}}_{2}^{-1}\mathbf{D}_{2})^{j-i}\bar{\mathbf{D}}_{2}^{-1}]$$

By substituting the above equation into Eq. (5), the matrix  $\Lambda^{-1}$  can be derived as

$$\Lambda^{-1} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{j} (2\alpha)^{k} {j \choose i}^{k} [\mathbf{V}_{1}^{T} (\bar{\mathbf{D}}_{1}^{-1} \mathbf{D}_{1})^{i} (\bar{\mathbf{D}}_{1}^{-1} \hat{\mathbf{D}}_{1})^{j-i} \bar{\mathbf{D}}_{1}^{-1} \mathbf{U}_{1}]^{k} \otimes [\mathbf{V}_{2}^{T} (\bar{\mathbf{D}}_{2}^{-1} \hat{\mathbf{D}}_{2})^{i} (\bar{\mathbf{D}}_{2}^{-1} \mathbf{D}_{2})^{j-i} \bar{\mathbf{D}}_{2}^{-1} \mathbf{U}_{2}]^{k}$$
(6)

Denote  $\mathbf{s} = \hat{\mathbf{D}}\hat{\mathbf{s}}$  and  $\hat{\mathbf{h}} = \hat{\mathbf{D}}^{-1}\mathbf{h}$ . Armed with the Kronecker product property  $\operatorname{vec}(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})\operatorname{vec}(\mathbf{B})$ , by substituting Eq. (6) into Eq. (4), we obtain the alignment matrix  $\mathbf{S} = \operatorname{mat}(\hat{\mathbf{D}}\hat{\mathbf{s}})$  as

$$\mathbf{S} = \alpha \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T + (1 - \alpha) \mathbf{H}$$
(7)

where **M** is an  $r_2 \times r_1$  matrix and is computed by

$$\mathbf{M} = (1-\alpha) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{j} 2^{k} \alpha^{k} {j \choose i}^{k} [\mathbf{V}_{2}^{T} (\bar{\mathbf{D}}_{2}^{-1} \hat{\mathbf{D}}_{2})^{i} (\bar{\mathbf{D}}_{2}^{-1} \mathbf{D}_{2})^{j-i} \bar{\mathbf{D}}_{2}^{-1} \mathbf{U}_{2}]^{k} \mathbf{V}_{2}^{T} \hat{\mathbf{H}} \mathbf{V}_{1} [\mathbf{U}_{1}^{T} (\bar{\mathbf{D}}_{1}^{-1} \mathbf{D}_{1})^{i} (\bar{\mathbf{D}}_{1}^{-1} \hat{\mathbf{D}}_{1})^{j-i} \bar{\mathbf{D}}_{1}^{-1} \mathbf{V}_{1}]^{k}$$
(8)

In practice, the prior knowledge matrix **H** is either low-rank (e.g., a rank-one uniform matrix) or very sparse. Having this in mind, we will mainly focus on how to learn the true low-rank structure part of **S** (i.e.,  $U_2MU_1$ ) from the input incomplete networks. This naturally leads to the following effective strategy. First, we temporarily treat the low-rank structure part as the alignment matrix to be solved in the optimization problems (i.e.,  $S \approx U_2MU_1$ ). After  $U_2$ , M,  $U_1$  are obtained, we can then calibrate the result by averaging between the learned **S** and the prior knowledge **H** to further emphasize the importance of the prior knowledge, i.e.,  $S \leftarrow (1 - \alpha)H + \alpha S$ . As we will show in the next section, a direct benefit of this strategy is that we can reduce the overall complexity (for both space and time cost) to be *linear*.

To take advantages of the low-rank structure of S under the above strategy, instead of minimizing Eq. (3) regarding the similarity matrix  $\hat{S}$ , we alternatively optimize the topology consistency

39:7



Fig. 2. Network completion via the alignment.

on the low-rank structure of alignment matrix  $S = U_2MU_1$  without the second regularization term, i.e., minimizing  $s^T(\hat{D} - A_1^* \otimes A_2^*)s$ . Given  $A_1^* = U_1V_1^T$ ,  $A_2^* = U_2V_2^T$ , by using the properties  $vec(A)^T vec(B) = Tr(A^TB)$  and  $vec(ABC) = (C^T \otimes A)vec(B)$ , network alignment across the complete networks can be formulated as minimizing the following objective function:

$$J_{2}(\mathbf{U}_{1}, \mathbf{V}_{1}, \mathbf{U}_{2}, \mathbf{V}_{2}, \mathbf{M}) = \frac{\gamma}{2} \mathbf{s}^{T} \operatorname{vec}(\mathbf{D}_{2} \mathbf{S} \mathbf{D}_{1} + \hat{\mathbf{D}}_{2} \mathbf{S} \hat{\mathbf{D}}_{1}) - \gamma \mathbf{s}^{T} \operatorname{vec}(\mathbf{U}_{2} \mathbf{V}_{2}^{T} \mathbf{S} \mathbf{V}_{1} \mathbf{U}_{1}^{T})$$

$$= \underbrace{\frac{\gamma}{2} \operatorname{Tr}(\mathbf{D}_{2} \mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T} \mathbf{D}_{1} \mathbf{U}_{1} \mathbf{M}^{T} \mathbf{U}_{2}^{T} + \hat{\mathbf{D}}_{2} \mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T} \hat{\mathbf{D}}_{1} \mathbf{U}_{1} \mathbf{M}^{T} \mathbf{U}_{2}^{T})}_{\text{alignment across complete networks}}$$

$$- \underbrace{\gamma \operatorname{Tr}(\mathbf{U}_{2} \mathbf{V}_{2}^{T} \mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T} \mathbf{V}_{1} \mathbf{U}_{1}^{T} \mathbf{U}_{1} \mathbf{M}^{T} \mathbf{U}_{2}^{T})}_{\text{i}} \tag{9}$$

alignment across complete networks

#### 3.2 Network Alignment Helps Network Completion.

Despite the effectiveness of the factorization-based network completion methods (i.e., Eq. (2)), in some applications, the information of a single network alone might be insufficient to correctly infer the missing edges. Meanwhile, the alignment across the two networks may provide extra hints of how to infer the missing edges. To be specific, since the aligned nodes are likely to share similar connectivity patterns, the observed existing edges in one network could potentially help recover the missing edges in the other network via the alignment matrix. Figure 2 presents an illustrative example. Here, node-*a* in  $G_1$  and node-*x* in  $G_2$  are aligned together, and the neighbor of *x* (say node-*y*) is aligned with the neighbor of *a* (e.g., node-*b*), which is not observed to connect with *a*. If we perform the completion solely based on the observed information of  $G_1$ , we might probably conclude that the edge between *a* and *b* does not exist. However, the facts that (1) *a* and *x* are aligned, (2) *b* and *y* are aligned, and (3) there is an edge between *x* and *y* might provide an *auxiliary confidence* about the existence of the edge between *a* and *b*. In general, we can estimate such auxiliary confidence of the existence of the edge between *a* and *b* in  $G_1$  as

$$\mathbf{A}_{1}^{*}(a,b) \approx \sum_{x,y}^{n_{2}} \mathbf{S}(a,x) \mathbf{S}(b,y) \mathbf{A}_{2}(x,y) = (\mathbf{S}^{T} \mathbf{A}_{2} \mathbf{S})(a,b)$$
(10)

where  $S = U_2 M U_1^T$  is the alignment matrix learned from the topology consistency.

In our experiments, we find that such auxiliary confidence is most powerful to estimate the existence/absence of an edge (a, b) when such an edge itself is not observed in  $\mathcal{G}_1$  (i.e.,  $(a, b) \in \overline{\Omega}_1$ ). Mathematically, this can be formulated as the following objective function.

$$J_{3}(\mathbf{U}_{1}, \mathbf{V}_{1}, \mathbf{U}_{2}, \mathbf{V}_{2}, \mathbf{M}) = \underbrace{\frac{\beta}{2} \|P_{\bar{\Omega}_{1}}(\mathbf{U}_{1}\mathbf{V}_{1}^{T} - \mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T})\|_{F}^{2}}_{\text{completion of }\mathcal{G}_{1} \text{ based on the observed edges in }\mathcal{G}_{2}} + \underbrace{\frac{\beta}{2} \|P_{\bar{\Omega}_{2}}(\mathbf{U}_{2}\mathbf{V}_{2}^{T} - \mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}\mathbf{A}_{1}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T})\|_{F}^{2}}$$

$$(11)$$

completion of  $\mathcal{G}_2$  based on the observed edges in  $\mathcal{G}_1$ 

where  $\bar{\Omega}_1$  and  $\bar{\Omega}_2$  are the unobserved set of  $A_1$  and  $A_2$ .

#### 3.3 Overall Objective Function

We impose the nonnegativity constraints on all the variables  $U_1$ ,  $V_1$ ,  $U_2$ ,  $V_2$ , M to guarantee that all the entries in matrices  $A_1^*$ ,  $A_2^*$ , S to be nonnegative. Combining Eq. (2), Eq. (9) and Eq. (11) together, the overall optimization problem is formulated as

$$\min_{\substack{U_1, V_1, U_2, V_2, M}} J(U_1, V_1, U_2, V_2, M) = J_1 + J_2 + J_3$$
s.t
$$U_1, U_2, V_1, V_2, M \ge 0$$
(12)

### 4 PROPOSED OPTIMIZATION ALGORITHM

In this section, we first present the proposed algorithm to solve the optimization problem in Eq. (12). Then, we analyze the proposed algorithm in terms of the correctness, the convergence and the complexity.

#### 4.1 Optimization Algorithm

Since the overall objective function Eq. (12) is not jointly convex, we optimize it by block coordinate descent. That is, the objective function is alternatively minimized with respect to one variable group (e.g.,  $U_1$ ) while fixing the others once at a time. For the sake of conciseness, we only show the minimization procedures over  $U_1$  and **M** in this section. Other variables such as  $V_1$ ,  $U_2$ ,  $V_2$  can be solved in a similar way and we put the details in Appendix A.

First, we show the update algorithm over  $U_1$ . The gradient of Eq. (2) with respect to  $U_1$  is computed by

$$\frac{\partial J_1}{\partial \mathbf{U}_1} = \mathbf{X}_1 - \mathbf{Y}_1 \tag{13}$$

where

$$\mathbf{X}_1 = [\mathbf{P}_{\Omega_1} \odot (\mathbf{U}_1 \mathbf{V}_1^T)] \mathbf{V}_1 + \lambda \mathbf{U}_1$$
$$\mathbf{Y}_1 = (\mathbf{P}_{\Omega_1} \odot \mathbf{A}_1) \mathbf{V}_1$$

and  $\mathbf{P}_{\Omega_1}(i, j) = 1$  for  $(i, j) \in \Omega_1$ , otherwise  $\mathbf{P}_{\Omega_1}(i, j) = 0$ .

As for Eq. (9), note that  $\mathbf{D}_1 = \text{diag}(\mathbf{U}_1\mathbf{V}_1^T\mathbf{1}_1)$  and  $\hat{\mathbf{D}}_1 = \text{diag}(\mathbf{1}_1^T\mathbf{U}_1\mathbf{V}_1^T)$  are also in terms of  $\mathbf{U}_1$ , thus the partial gradient is computed by

$$\frac{\partial J_2}{\partial \mathbf{U}_1} = \mathbf{X}_2 - \mathbf{Y}_2 \tag{14}$$

where

$$\begin{aligned} \mathbf{X}_{2} &= \frac{\gamma}{2} [ (\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{D}_{2}\mathbf{U}_{2}\mathbf{M}) \odot \mathbf{U}_{1} ] \mathbf{1}_{r_{1}}\mathbf{1}_{1}^{T}\mathbf{V}_{1} + \frac{\gamma}{2}\mathbf{1}_{1}\mathbf{1}_{r_{1}}^{T} [ (\mathbf{M}^{T}\mathbf{U}_{2}^{T}\hat{\mathbf{D}}_{2}\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}) \odot \mathbf{U}_{1}^{T} ] \mathbf{V}_{1} \\ &+ \gamma (\mathbf{D}_{1}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{D}_{2}\mathbf{U}_{2}\mathbf{M} + \hat{\mathbf{D}}_{1}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\hat{\mathbf{D}}_{2}\mathbf{U}_{2}\mathbf{M}) \\ \mathbf{Y}_{2} &= \gamma \mathbf{V}_{1}\mathbf{U}_{1}^{T}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{U}_{2}\mathbf{V}_{2}^{T}\mathbf{U}_{2}\mathbf{M} + \gamma \mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{U}_{2}\mathbf{V}_{2}^{T}\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}\mathbf{V}_{1} + \gamma \mathbf{U}_{1}\mathbf{V}_{1}^{T}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{V}_{2}\mathbf{U}_{2}^{T}\mathbf{U}_{2}\mathbf{M} \end{aligned}$$

And the gradient of Eq. (11) over  $U_1$  is

$$\frac{\partial J_3}{\partial \mathbf{U}_1} = \mathbf{X}_3 - \mathbf{Y}_3 \tag{15}$$

where

$$\begin{split} \mathbf{X}_{3} &= 2\beta [\mathbf{P}_{\bar{\Omega}_{1}} \odot (\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T})]\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M} + \beta [\mathbf{P}_{\bar{\Omega}_{1}} \odot (\mathbf{U}_{1}\mathbf{V}_{1}^{T})]\mathbf{V}_{1} \\ &+ 2\beta \mathbf{A}_{1}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T} [\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}\mathbf{A}_{1}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T})]\mathbf{U}_{2}\mathbf{M} \\ \mathbf{Y}_{3} &= \beta [\mathbf{P}_{\bar{\Omega}_{1}} \odot (\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T})]\mathbf{V}_{1} + \beta [\mathbf{P}_{\bar{\Omega}_{1}} \odot (\mathbf{U}_{1}\mathbf{V}_{1}^{T} + \mathbf{V}_{1}\mathbf{U}_{1}^{T})]\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M} \\ &+ \beta \mathbf{A}_{1}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T} [\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{V}_{2}^{T} + \mathbf{V}_{2}\mathbf{U}_{2}^{T})]\mathbf{U}_{2}\mathbf{M} \end{split}$$

and matrix  $\mathbf{P}_{\bar{\Omega}_2}(i, j) = 1$  for any  $(i, j) \notin \Omega_2$ . A fixed-point solution of  $\frac{\partial J}{\partial \mathbf{U}_1} = \mathbf{0}$  under the nonnegativity constraint of  $\mathbf{U}_1$  leads to the following multiplicative update rule

$$\mathbf{U}_{1}(p,q) \leftarrow \mathbf{U}_{1}(p,q) \sqrt[4]{\frac{\mathbf{Y}_{1}(p,q) + \mathbf{Y}_{2}(p,q) + \mathbf{Y}_{3}(p,q)}{\mathbf{X}_{1}(p,q) + \mathbf{X}_{2}(p,q) + \mathbf{X}_{3}(p,q)}}$$
(16)

Second, the optimization algorithm over M is given as below. The gradient of Eq. (9) w.r.t M can be derived as

$$\frac{\partial J_2}{\mathbf{M}} = \mathbf{X}_4 - \mathbf{Y}_4 \tag{17}$$

where

$$\mathbf{X}_4 = \gamma \mathbf{U}_2^T \mathbf{D}_2 \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \mathbf{D}_1 \mathbf{U}_1 + \gamma \mathbf{U}_2^T \hat{\mathbf{D}}_2 \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \hat{\mathbf{D}}_1 \mathbf{U}_1$$
  
$$\mathbf{Y}_4 = \gamma \mathbf{U}_2^T \mathbf{U}_2 \mathbf{V}_2^T \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \mathbf{V}_1 \mathbf{U}_1^T \mathbf{U}_1 + \gamma \mathbf{U}_2^T \mathbf{V}_2 \mathbf{U}_2^T \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \mathbf{U}_1 \mathbf{V}_1^T \mathbf{U}_1$$

And the gradient of Eq. (11) w.r.t M is computed by

$$\frac{\partial J_3}{\mathbf{M}} = \mathbf{X}_5 - \mathbf{Y}_5 \tag{18}$$

where

$$\begin{aligned} \mathbf{X}_5 &= \beta \mathbf{U}_2^T \mathbf{A}_2 \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T [\mathbf{P}_{\bar{\Omega}_1} \odot (\mathbf{U}_1 \mathbf{V}_1^T + \mathbf{V}_1 \mathbf{U}_1^T)] \mathbf{U}_1 \\ &+ \beta \mathbf{U}_2^T [\mathbf{P}_{\bar{\Omega}_2} \odot (\mathbf{U}_2 \mathbf{V}_2^T + \mathbf{V}_2 \mathbf{U}_2^T)] \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \mathbf{U}_1^T \mathbf{A}_1 \mathbf{U}_1 \\ \mathbf{Y}_5 &= 2\beta \mathbf{U}_2^T \mathbf{A}_2 \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T [\mathbf{P}_{\bar{\Omega}_1} \odot (\mathbf{U}_1 \mathbf{M}^T \mathbf{U}_2^T \mathbf{A}_2 \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T)] \mathbf{U}_1 \\ &+ 2\beta \mathbf{U}_2^T [\mathbf{P}_{\bar{\Omega}_2} \odot (\mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \mathbf{A}_1 \mathbf{U}_1 \mathbf{M}^T \mathbf{U}_2^T)] \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \mathbf{A}_1 \mathbf{U}_1 \end{aligned}$$

Consequently, the fixed-point solution of  $\frac{\partial J}{\partial M} = 0$  under the nonnegative constraint leads to the following update rule

$$\mathbf{M}(p,q) \leftarrow \mathbf{M}(p,q) \sqrt[4]{\frac{\mathbf{Y}_4(p,q) + \mathbf{Y}_5(p,q)}{\mathbf{X}_4(p,q) + \mathbf{X}_5(p,q)}}$$
(19)

ACM Trans. Web, Vol. 9, No. 4, Article 39. Publication date: December 2018.

39:10

Algorithm 1 INEAT: Incomplete Network Alignment.

- **Input:** (1) the adjacency matrices  $A_1$ ,  $A_2$  of the incomplete networks  $\mathcal{G}_1$ ,  $\mathcal{G}_2$ , (2) the optional prior alignment preference H, (3) the rank sizes  $r_1, r_2$ , (3) the parameters  $\alpha, \lambda, \gamma, \beta$ . and (4) the maximum iteration number  $t_{\text{max}}$ .
- **Output:** (1) the alignment matrix S between  $\mathcal{G}_1$  and  $\mathcal{G}_2$ , and (2) the complete adjacency matrices  $A_{1}^{*}, A_{2}^{*}.$ 
  - 1: Initialize  $U_1, V_1, U_2, V_2$  by Eq. (20), M by Eq. (21), t = 1;
  - 2: while not converge and  $t \leq t_{\text{max}}$  do
  - Update  $U_1$  by Eq. (16); 3:
  - Update  $V_1$  by Eq. (27); 4:
  - Update  $U_2$  by Eq. (28); 5:
  - Update  $V_2$  by Eq. (29); 6:
  - Update M by Eq. (19); 7:
  - Set  $t \leftarrow t + 1$ ; 8:
  - 9: end while
- 10:  $\mathbf{A}_1^* = \mathbf{U}_1 \mathbf{V}_1^T$  and  $\mathbf{A}_2^* = \mathbf{U}_2 \mathbf{V}_2^T$ . 11:  $\mathbf{S} = \alpha \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T + (1 \alpha) \mathbf{H}$ .

Initialization. Since the optimization problem in Eq. (12) is not a joint convex problem, a good initialization of each variable group could play an important role of obtaining a good final solution. For  $U_1$  and  $U_2$ , we initialize them by solving the symmetric nonnegative matrix factorization of  $A_1$ and  $\mathbf{A}_2$ , e.g., minimizing  $\|\mathbf{A}_1 - \mathbf{U}_1\mathbf{U}_1^T\|_F^2$  over  $\mathbf{U}_1 \ge \mathbf{0}$ . Same as [9], we use the following multiplicative update rule to obtain the solution

$$\mathbf{U}_1 \leftarrow \mathbf{U}_1 \odot \left[1 - \epsilon + \epsilon \frac{\mathbf{A}_1 \mathbf{U}_1}{\mathbf{U}_1 (\mathbf{U}_1^T \mathbf{U}_1)}\right]$$
(20)

where  $\epsilon$  is suggested to be set to 0.5 in practice. Then we set  $V_1 = U_1$  due to the symmetry of  $A_1$ and initialize  $U_2$ ,  $V_2$  similarly. As for the variable M, given the initial  $U_1 = V_1$ ,  $U_2 = V_2$ , we can simplify the computation of Eq. (8) and initialize M as

$$\mathbf{M} = (1 - \alpha) \sum_{k=0}^{K} \alpha^{k+1} (\mathbf{U}_{2}^{T} \mathbf{D}_{2}^{-1} \mathbf{U}_{2})^{k} \mathbf{U}_{2}^{T} \mathbf{D}_{2}^{-1} \mathbf{H} \mathbf{D}_{1}^{-1} \mathbf{U}_{1} (\mathbf{U}_{1}^{T} \mathbf{D}_{1}^{-1} \mathbf{U}_{1})^{k}$$
(21)

where the constant *K* can be set to a relatively large number, e.g., 100.

Overall, the proposed algorithm is summarized in Algorithm 1. First, it initializes each variable as line 1. Then, the algorithm alternatively updates each variable group one by one (line 3-7) until it converges or the maximum iteration number  $t_{max}$  is reached. The algorithm finally outputs the complete networks  $A_1^*$ ,  $A_2^*$  (line 10), and the alignment matrix S as line 11.

#### 4.2 **Proof and Analysis**

In this subsection, we provide the theoretical analysis of the updating rule of  $U_1$ . We first prove that the fixed-point solution of Eq. (16) satisfies the KKT condition. Then we analyze its convergence, as well as its time and space complexity. The analyses and proofs for other variables are similar and are omitted in the paper for brevity.

THEOREM 1. Correctness of Eq. (16). At convergence, the fixed-point solution of Eq. (16) satisfies the KKT condition.

**PROOF.** Let  $\Sigma \in \mathbb{R}^{n_1 \times r_1}$  be the Lagrangian multiplier and the Lagrangian function of Eq. (12) be

$$L(\mathbf{U}_1) = J(\mathbf{U}_1) - \mathrm{Tr}(\boldsymbol{\Sigma}^T \mathbf{U}_1)$$

By setting the gradient of L w.r.t U<sub>1</sub> to 0, we obtain

$$\Sigma = X_1 + X_2 + X_3 - Y_1 - Y_2 - Y_3$$
(22)

The KKT complementary condition for the nonnegativity of U1 gives

$$(X_1 + X_2 + X_3 - Y_1 - Y_2 - Y_3) \odot U_1 = \mathbf{0}$$
(23)

According to the updating rule Eq. (16), at convergence, we have for  $\forall p, q$ ,

$$\mathbf{U}_{1}(p,q) = \mathbf{U}_{1}(p,q) \sqrt[4]{\frac{\mathbf{Y}_{1} + \mathbf{Y}_{2} + \mathbf{Y}_{3}}{\mathbf{X}_{1} + \mathbf{X}_{2} + \mathbf{X}_{3}}}$$
(24)

which is equivalent to

$$(X_1 + X_2 + X_3 - Y_1 - Y_2 - Y_3) \odot (U_1)^4 = \mathbf{0}$$
(25)

Eq. (23) and Eq. (25) are equivalent, so at convergence, Eq. (16) satisfies the KKT condition.  $\Box$ 

Then, we show the convergence of updating  $U_1$  under Eq. (16). First, the following lemma gives the auxiliary function for the objective function Eq. (12) w.r.t  $U_1$ .

LEMMA 2. Auxiliary function of  $J(U_1)$ . Let  $J(U_1)$  denote all the terms in Eq. (12) that contains  $U_1$ , then the following function  $Z(U_1, \tilde{U}_1)$  $U_1^{4}(p, q) + \tilde{U}^{4}(p, q)$ 

$$Z(\mathbf{U}_{1},\tilde{\mathbf{U}}_{1}) = \underbrace{\frac{1}{4} \sum_{p,q} \left[ (\mathbf{P}_{\Omega_{1}} \odot (\tilde{\mathbf{U}}_{1}\mathbf{V}_{1}^{T}))\mathbf{V}_{1} \right](p,q) \underbrace{\frac{\mathbf{U}_{1}(p,q) + \mathbf{U}_{1}(p,q)}{\tilde{\mathbf{U}}_{1}^{3}(p,q)}}_{T_{1}'} - \underbrace{\sum_{p,q} \left[ (\mathbf{P}_{\Omega_{1}} \odot \mathbf{A}_{1})\mathbf{V}_{1} \right](p,q) \underbrace{\tilde{\mathbf{U}}_{1}(p,q)(1 + \log \frac{\mathbf{U}_{1}(p,q)}{\tilde{\mathbf{U}}_{1}(p,q)})}_{T_{2}'} + \underbrace{\frac{\lambda}{4} \sum_{p,q} \tilde{\mathbf{U}}_{1}(p,q) \frac{\mathbf{U}_{1}^{4}(p,q) + \tilde{\mathbf{U}}_{1}^{4}(p,q)}{\tilde{\mathbf{U}}_{1}^{3}(p,q)}}_{T_{3}'} + \underbrace{\frac{\lambda}{4} \sum_{p,q} \left[ (\mathbf{P}_{\Omega_{1}} \odot (\tilde{\mathbf{U}}_{1}\mathbf{V}_{1}^{T}))\mathbf{V}_{1} \right](p,q) \frac{\mathbf{U}_{1}^{4}(p,q) + \underbrace{Y}_{12} \sum_{p,q} \mathbf{Z}_{1}(p,q) \frac{\mathbf{3U}_{1}^{4}(p,q) + \widetilde{\mathbf{U}}_{1}^{4}(p,q)}{\widetilde{\mathbf{U}}_{1}^{3}(p,q)}}_{T_{3}'} + \underbrace{\frac{\beta}{2} \sum_{p,q} \left[ (\mathbf{P}_{\Omega_{1}} \odot (\tilde{\mathbf{U}}_{1}\mathbf{V}_{1}^{T}))\mathbf{V}_{1} \right](p,q) \frac{\mathbf{U}_{1}^{4}(p,q) + \underbrace{\tilde{\mathbf{U}}_{1}^{4}(p,q)}{\underbrace{\tilde{\mathbf{U}}_{1}^{3}(p,q)}}_{T_{6}'}}_{T_{6}'} + \underbrace{\frac{\beta}{2} \sum_{p,q} \mathbf{Z}_{2}(p,q) \frac{\mathbf{U}_{1}^{4}(p,q)}{\underbrace{\tilde{\mathbf{U}}_{1}^{3}(p,q)}}_{T_{1}'} + \underbrace{\frac{\beta}{2} \sum_{p,q} \mathbf{Z}_{3}(p,q) \frac{\mathbf{U}_{1}^{4}(p,q)}{\underbrace{\tilde{\mathbf{U}}_{1}^{3}(p,q)}}_{T_{6}'} + \beta T_{10}' + \beta T_{10}'$$

where

ACM Trans. Web, Vol. 9, No. 4, Article 39. Publication date: December 2018.

39:12

$$\begin{split} \mathbf{Z}_{1} &= \frac{1}{2} \big[ (\tilde{\mathbf{U}}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{D}_{2}\mathbf{U}_{2}\mathbf{M}) \odot \tilde{\mathbf{U}}_{1} \big] \mathbf{1}_{r_{1}} \mathbf{1}_{1}^{T}\mathbf{V}_{1} + \frac{1}{2} \mathbf{1}_{1} \mathbf{1}_{r_{1}}^{T} \big[ (\mathbf{M}^{T}\mathbf{U}_{2}^{T}\hat{\mathbf{D}}_{2}\mathbf{U}_{2}\mathbf{M}\tilde{\mathbf{U}}_{1}) \odot \tilde{\mathbf{U}}_{1} \big] \mathbf{V}_{1} \\ &+ diag (\tilde{\mathbf{U}}_{1}\mathbf{V}_{1}^{T}\mathbf{1}_{1}) \tilde{\mathbf{U}}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{D}_{2}\mathbf{U}_{2}\mathbf{M} + diag (\mathbf{V}_{1}\mathbf{U}_{1}^{T}\mathbf{1}_{1}) \tilde{\mathbf{U}}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\hat{\mathbf{D}}_{2}\mathbf{U}_{2}\mathbf{M} \\ \mathbf{Z}_{2} &= [\mathbf{P}_{\tilde{\Omega}_{1}} \odot (\tilde{\mathbf{U}}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M}\tilde{\mathbf{U}}_{1}^{T}) ] \tilde{\mathbf{U}}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M} \\ \mathbf{Z}_{3} &= \mathbf{A}_{1}\tilde{\mathbf{U}}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T} [\mathbf{P}_{\tilde{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{M}\tilde{\mathbf{U}}_{1}^{T}\mathbf{A}_{1}\tilde{\mathbf{U}}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}] \mathbf{U}_{2}\mathbf{M} \\ \mathbf{T}_{5}' &= -\sum_{o,p,q,r,s} (\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{U}_{2}\mathbf{V}_{2}^{T}\mathbf{U}_{2}\mathbf{M})(o,q)\mathbf{V}_{1}(p,r)\tilde{\mathbf{U}}_{1}(s,r)\tilde{\mathbf{U}}_{1}(s,o) \times \tilde{\mathbf{U}}_{1}(p,q) \\ &(1 + \log\frac{\mathbf{U}_{1}(p,q)\mathbf{U}_{1}(s,r)\mathbf{U}_{1}(s,o)}{\tilde{\mathbf{U}}_{1}(p,q)\tilde{\mathbf{U}}_{1}(s,r)\tilde{\mathbf{U}}_{1}(s,o)} ) \\ \mathbf{T}_{9}' &= -\sum_{o,p,q,r,s} \mathbf{P}_{\tilde{\Omega}_{1}}(p,o)\mathbf{V}_{1}(o,q)(\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M})(s,r)\tilde{\mathbf{U}}_{1}(o,s) \times \tilde{\mathbf{U}}_{1}(p,r)\tilde{\mathbf{U}}_{1}(p,q) \\ &(1 + \log\frac{\mathbf{U}_{1}(p,q)\mathbf{U}_{1}(o,s)\mathbf{U}_{1}(p,r)}{\tilde{\mathbf{U}}_{1}(p,q)\tilde{\mathbf{U}}_{1}(o,s)\tilde{\mathbf{U}}_{1}(p,r)} ) \\ \mathbf{T}_{10}' &= -\sum_{o,p,q,r,s,t} \mathbf{P}_{\tilde{\Omega}_{2}}(p,q)(\mathbf{U}_{2}\mathbf{V}_{2}^{T})(p,q)(\mathbf{U}_{2}\mathbf{M})(p,r)\tilde{\mathbf{U}}_{1}(s,r)\mathbf{A}_{1}(s,t)\tilde{\mathbf{U}}_{1}(t,o)(\mathbf{M}^{T}\mathbf{U}_{2}^{T})(o,q) \\ &\times (1 + \log\frac{\mathbf{U}_{1}(s,r)\mathbf{U}_{1}(t,o)}{\tilde{\mathbf{U}}_{1}(s,r)\tilde{\mathbf{U}}_{1}(t,o)}) \end{aligned}$$

is an auxiliary function of  $J(U_1)$  for any  $U_1, \tilde{U}_1 \ge 0$  after removing some constant terms such that  $Z(U_1, \tilde{U}_1) \ge J(U_1)$  and  $Z(U_1, U_1) = J(U_1)$ . And it is also a convex function w.r.t  $U_1$  and its global minima is

$$\mathbf{U}_{1}(p,q) = \tilde{\mathbf{U}}_{1}(p,q) \sqrt[4]{\frac{\tilde{\mathbf{Y}}_{1}(p,q) + \tilde{\mathbf{Y}}_{2}(p,q) + \tilde{\mathbf{Y}}_{3}(p,q)}{\tilde{\mathbf{X}}_{1}(p,q) + \tilde{\mathbf{X}}_{2}(p,q) + \tilde{\mathbf{X}}_{3}(p,q)}}$$
(26)

where  $\tilde{X}_i, \tilde{Y}_i, i = 1, 2, 3$  are all in terms of  $\tilde{U}_1$  while sharing the same formulas with  $X_i, Y_i, i = 1, 2, 3$ .

PROOF. Refer the proof to Appendix B.

Next, we show the convergence of updating  $U_1$  by Eq. (16) in the following theorem.

THEOREM 2. Convergence of Eq. (16). When other variables are fixed, under the updating rule Eq. (16), the objective function w.r.t  $U_1$  monotonically non-increases.

**PROOF.** Denote  $\mathbf{U}_1$  at iteration *t* as  $\mathbf{U}_1^{(t)}$ . According to Lemma 2, the global minima  $\mathbf{U}_1^{(t+1)}$  of the auxiliary function is achieved by minimizing the function  $Z(\mathbf{U}_1, \mathbf{U}_1^{(t)})$  over  $\mathbf{U}_1$ , which leads to

$$Z(\mathbf{U}_{1}^{(t+1)},\mathbf{U}_{1}^{(t)}) \le Z(\mathbf{U}_{1}^{(t)},\mathbf{U}_{1}^{(t)}) = J(\mathbf{U}_{1}^{(t)})$$

Besides, based on Lemma 2,  $J(\mathbf{U}_1^{(t+1)}) \leq Z(\mathbf{U}_1^{(t+1)}, \mathbf{U}_1^{(t)})$  and therefore  $J(\mathbf{U}_1^{(t+1)}) \leq J(\mathbf{U}_1^{(t)})$  which means the objective function w.r.t.  $\mathbf{U}_1$  is monotonically non-increasing.

The time and space complexity of each updating iteration in Algorithm 1 are summarized in Lemma 3. Note that by exploring the low-rank structure of the alignment matrix, the time complexity is reduced to *linear*.

LEMMA 3. Complexity of INEAT. The time complexity of each update iteration in Algorithm 1 is  $O(nr^2 + min\{|\bar{\Omega}|, |\Omega|\}r)$ , and the space complexity is  $O(nr + min\{|\bar{\Omega}|, |\Omega|\})$  where  $n, |\Omega|, |\bar{\Omega}|$  are the number of nodes, the number of observed and unobserved entries in two incomplete networks respectively. And r denotes the rank of networks.

PROOF. For time complexity, calculating the term  $X_1, Y_1, X_3$  and  $Y_3$  in each iteration has  $O(nr^2 + mr + \min\{|\bar{\Omega}|, |\Omega|\}r)$  time complexity and O(m+nr) space complexity. Note that  $P_{\Omega_1} + P_{\bar{\Omega}_1} = \mathbf{1}_{n_1 \times n_1}$ . In this way, for example,  $X_1$  can be computed from either  $P_{\Omega_1}$  or  $P_{\bar{\Omega}_1}$ . Thus, we use  $\min\{|\bar{\Omega}|, |\Omega|\}$  for the complexity analysis. For terms  $X_2$  and  $Y_2$ , it takes  $O(nr^2)$  time complexity and O(nr) space complexity. For other variables  $V_1, U_2, V_2$  and M, since the analyses are similar, we omit the analyses for brevity. Overall, we can obtain the time and space complexity in the above lemma.

We remark that the linear complexity is obtained in each updating iteration of Algorithm 1. If we carry out line 10-11 in a straightforward way, it will incur an additional  $O(n^2)$  cost due to the multiplications between low-rank matrices (e.g.,  $\mathbf{U}_1\mathbf{V}_1^T$ ,  $\mathbf{U}_2\mathbf{M}\mathbf{U}_1$ , etc.) as well as the need to store the potentially dense matrices (e.g.,  $\mathbf{A}_1^*$ ,  $\mathbf{S}$ , etc). To address this issue, we can store the resulting  $\mathbf{A}_1^*$ ,  $\mathbf{A}_2^*$  and  $\mathbf{S}$  in a compact way by the corresponding low-rank matrices. Then when we access a certain entry of the matrix (e.g.  $\mathbf{A}_1^*$ ), we perform the vector-vector inner product between the corresponding rows of  $\mathbf{U}_1$  and  $\mathbf{V}_1$ .

## 5 EXPERIMENTAL RESULTS

In this section, we present the experimental results of the proposed algorithm INEAT. We evaluate our algorithm in the following two aspects:

- *Effectiveness*: How accurate is our algorithm for aligning incomplete networks? How effective is our algorithm to recover missing edges by leveraging the alignment result?
- Efficiency: How fast and scalable is our algorithm?

## 5.1 Experimental Setup

**Datasets.** We evaluate the proposed algorithm on three types of real-world networks, including the collaboration network, infrastructure network and social networks. The statistics of all the datasets are summarized in Table 2.

- *Collaboration Network*: We use the collaboration network in the general relativity and quantum cosmology (*Gr-Qc*) area from the e-print arXiv [18]. In the network, each node represents an author and there exists an edge if two authors have co-authored at least one paper.
- *Infrastructure Network*: This dataset is a network of Autonomous Systems (AS) inferred from Oregon route-view [18]. In the network, nodes are the routers, and edges represent the peering information among routers.
- *Social Network*: We use the social network collected from Google+ [19]. In the network, nodes are the users and an edge denotes that one user has the other user in her circles. We also use the Youtube network [39] where nodes are the Youtube users and edges represent the friendship among users.
- *Gordian Networks*: This dataset contains communication networks via different channels. In particular, we aim to align the communication networks via phone (Channel 1) and emails (Channel 2). Each node in both networks represents a person. An edge in Channel 1 network indicates two people contact each other through phone whereas each edge in Channel 2 network represents two people send an email. There are 1,000 common nodes in both networks that are used as the alignment ground-truth.

Based on these datasets except Gordian dataset, we construct four pairs of incomplete networks for alignment evaluations by the following steps. For each dataset, we first generate a random permutation matrix and use it to construct the second (permuted) network. Then, in each of these two networks, we remove 0.1%, 0.5%, 1%, 5%, 10%, 15%, 20% of the total number of edges uniformly at random to generate the unobserved edges. For the Gordian dataset (Channel 1 and Channel 2),

Category	Network	# of Nodes	# of Edges
Collaboration	Gr-Qc	5,241	14,484
Infrastructure	Oregon	7,352	15,665
Social	Google+	23,628	39,194
Social	Youtube	1,134,890	2,987,624
Communication	Gordian Channel 1	1,000	41,191
Communication	Gordian Channel 2	1,003	4,627

Table 2. Statistics of Datasets.



(a) Gr-Qc collaboration (b) Oregon infrastructure (c) Google+ social net-(d) Gordian channel 1 vs. networks. networks. channel 2.

Fig. 3. (Higher is better.) Alignment accuracy vs. the number of unobserved edges in the networks.

we first compute the edge betweenness score for each edge, which is sum of the fraction of all pairs of shortest paths through the edge [6]. Then we normalize the edge betweenness scores such that they sum to 1, i.e.,  $\sum_{(u,v) \in \mathcal{G}_1} s_b((u,v)) = 1$  where  $s_b((u,v))$  is the edge betweenness score of edge (u, v) and then we use the normalized scores as the probabilities to remove the corresponding edge as an unobserved edge. We run our algorithm and other comparison methods in all the pairs of incomplete networks.

# **Comparison Methods.**

- *Alignment*. To evaluate the alignment performance of our proposed algorithm INEAT<sup>3</sup>, we compare it with the following existing network alignment algorithms, including (1) *NetAlign* [3], (2) *IsoRank* [34], (3) *FINAL-P+* [43]. Besides, in order to validate whether alignment and imputation are mutually beneficial from each other, we use the low-rank networks completed solely by Eq. (2) as the input networks for *FINAL-P+*. We name this method as *FINAL-IMP*. We also show the alignment results by the degree similarity (*DegSim*), which is also used as the prior knowledge matrix **H** of INEAT.
- *Completion.* To evaluate the completion performance, we compare our algorithm with the existing matrix completion methods which are for the single network completion task, including (1) a matrix factorization method based on Eq. (2) (*NMF-IMP*), (2) an accelerated proximal gradient based nuclear norm minimization method (*NNLS*) [36], (3) a Riemannian trust-region based matrix completion method (*RTRMC*) [5].

**Machines and Repeatability.** All experiments are performed on a Windows machine with four 3.6GHz Intel Cores and 32G RAM. The algorithms are programmed with MATLAB using a single thread.

<sup>&</sup>lt;sup>3</sup>The code can be found in http://www.public.asu.edu/~szhan172/ineat.zip



Fig. 4. (Higher is better.) Recovery rate vs. the number of unobserved edges in the networks.

### 5.2 Effectiveness Analysis

We first evaluate the alignment accuracy with different numbers of unobserved edges in the incomplete networks. We use a heuristic greedy matching algorithm as the post processing step on the alignment matrix to obtain the one-to-one mapping matrix between two input networks, then compute the alignment accuracy with respect to the ground-truth (i.e., the permutation matrix). The results are summarized in Figure 3. We have the following observations. First, we observe that INEAT outperforms the baseline methods with different numbers of unobserved edges. To be specific, our method achieves an up to 30% alignment accuracy improvement, compared with the baseline methods that directly align across two incomplete networks (i.e., NetAlign, IsoRank, FINAL-P+). Second, the degree similarity (i.e., H) alone gives a very poor performance on the alignment accuracy, whereas by averaging H and  $U_2MU_1$ , the alignment matrix (i.e., results of INEAT) provides a much better accuracy. This verifies the effectiveness of our strategy combining the low-rank structure of alignment matrix and prior knowledge H. Third, the accuracy of INEAT is higher than that of FINAL-IMP, which indicates that solving the alignment and imputation tasks simultaneously indeed achieves a better performance than the *completion-then-alignment* strategy. Specifically, as Figure 3 (a) and Figure 3 (b) show, in some cases, the pure completion may introduce too much noise in the incomplete networks and hence lead to an even worse alignment result than that of other alignment baseline methods (those without performing network completion at all).

Second, to evaluate the effectiveness of INEAT for network completion, we assume the missing edges are recovered if the corresponding entries of the completed adjacency matrix are larger than a certain threshold (e.g., set to be 0.3 in our paper). Then, we calculate the recovery rate over the total number of missing edges. In addition, as the algorithms of network completion may achieve different performance with different initializations, we repeat the algorithms for 20 runs and present the mean edge recovery rates and variances. The results are shown in Figure 4. As we can see, INEAT has a higher recovery rate than other baseline methods, indicating that the completion performance is indeed improved by leveraging the alignment across two networks. Besides, the network completion performance of our algorithms are not sensitive to the algorithm initializations. For *NNLS* and *RTRMC* baseline methods, we did not observe any variances.

Third, we study how different parameters affect the alignment accuracy. In our experiments, we mainly study three parameters, including (1)  $\gamma$  which controls the importance of alignment task, (2)  $\beta$  which controls the importance of cross-network completion task, and (3) r which is the rank of the complete network. The results are shown in Figure 5. As we can see, the alignment accuracy



(a) Alignment accuracy vs.(b) Alignment accuracy vs.(c) Alignment accuracy vs.(d) Alignment accuracy vs.  $\gamma$ .  $\beta$ . rank r.  $\alpha$ .

Fig. 5. Parameter study on collaboration networks with 5% unobserved edges: study the effect of the parameters  $\gamma$ ,  $\beta$ , rank r and  $\alpha$  in terms of alignment accuracy.

is stable within a wide range of parameter settings. Besides, Figure 5 (c) suggests that a relatively small rank might be sufficient to achieve a satisfactory alignment performance. We also observe in Figure 5 (d) that (1) by leveraging the combination of both  $\mathbf{U}_2\mathbf{M}\mathbf{U}_1^T$  and the prior information H can significantly improve the alignment performance, and (2)  $\alpha = 0.5$  leads to better results in most cases.

#### 5.3 Efficiency Analysis

**Quality-Speed Trade-off.** In order to evaluate the trade-off between the effectiveness and efficiency of our method, we measure the quality from two aspects, including the quality of alignment and that of network completion. Here, we show the trade-off results on the collaboration network with 10% unobserved edges in Figure 6. As we can see in Figure 6 (a), the running time of our method INEAT is slightly higher than *IsoRank* and *FINAL-P+*, but it achieves a 15%-25% alignment accuracy improvement across the incomplete networks. Meanwhile, our method is much faster than *NetAlign*.

On the other hand, to evaluate the quality of network completion, note that the running time is the time for completing two incomplete networks. As Figure 6 (b) shows, INEAT obtains a better recovery rate and less running time. To be specific, compared with *NMF-IMP*, INEAT can recover 10% more missing edges with a similar running time. Besides, INEAT achieves a slightly better recovery rate and a much faster speed than *NNLS* and *RTRMC*.

**Scalability.** We use the largest dataset (Youtube) to study the scalability of our proposed method INEAT (i.e., running time vs. size of the network). Here, we use the same method to extract and construct several pairs of incomplete subgraphs with different sizes from the entire network. As we can see from Figure 7, the running time of the algorithm is *linear* w.r.t the number of nodes in the networks which is consistent with our time complexity analysis.

#### 6 RELATED WORK

**Network Alignment.** In general, network alignment has two categories, i.e., local alignment and global alignment. Among others, local network alignment aims to uncover the alignment among small regions across multiple networks, such as motifs, small subgraphs and so on. Some recent works include [4, 26, 30]. Nevertheless, local network alignment might be too restrictive to effectively find the node correspondence. On the other side, many global network alignment algorithms that targets to find node alignment, are based on the *topology consistency*. For example, one early well-known approach *IsoRank* computes the cross-network pairwise topology similarities by propagating the similarities of their neighboring node pairs and it is shown that this can be formulated as a random-walk propagation procedure in the Kronecker product graph [34]. In



(a) Balance between running time and align-(b) Balance between running time and rement accuracy. covery rate.

Fig. 6. Quality-speed results on the collaboration network with 10% unobserved edges.



Fig. 7. Running time vs. the number of nodes in the networks.

addition, *IsoRankN* [20] extends the original *IsoRank* algorithm by using PageRank-Nibble [1] to align multiple networks. *BigAlign* [16] and *UMA* [42] both assume that one network is a noisy permutation of the other network, whereas [42] is generalized to align multiple networks by adding the *transitivity* constraints. *NetAlign* formulates the network alignment problem as an optimization problem to maximize the number of aligned neighboring node pairs [3] and solve it based on a belief-propagation heuristic.

More recently, Liu et al. propose an algorithm that learns the embedding of the nodes while making the aligned nodes are closed to each other in the embedding space [23]. Then the distances among the embedding vectors of nodes are used to measure the probabilities that nodes are aligned. Another recent embedding-based alignment method for attributed networks is proposed in [14]. However, these embedding-based methods implicitly suffer from the space disparity. To address this issue, Du et al. propose an embedding based method MrMine that forces the embedding vectors of different objects of the networks at different resolutions to lie in the same space [11]. In addition, Zhang et al. propose to mitigate this issue by postprocessing the node embedding vectors by a non-rigid point-set registration [45]. Moreover, Vijayan et al. proposes MAGNA++ [38] that

simultaneously maximizes the node conservation and edge conservation which are widely used in bioinformatics. Other network alignment algorithms proposed in the bioinformatics community include *HubAlign* [13], *Natelie* [12], *L-GRAAL* [25], etc. However, most of these methods only assume the *global consistency* in the network topology and possibly leverage the attribute information by calculating the attribute similarity matrix as the prior alignment matrix. One potential drawback of these methods is not considering the consistency in the attributes of the networks.

Some early works in another relevant topic (i.e., anchor link prediction), which use both structural information and attribute information to map users across networks, include [15, 40]. A recent work *COSNET* formulates the local consistency among the attributes of each node and the global topology consistency, as well as the transitivity property into an energy-based model to find the alignment across multiple attributed networks [46]. However, these methods are all supervised and require the exact alignment relationships as the training data. On the other side, Zhang et al. propose an attributed network alignment algorithm by adopting both the topological and attribute consistency principles [43]. This work formulates these consistency principles into a convex quadratic problem and propose a fixed-point solution to solve it. Du et al. an efficient solver based on Krylov subspace to accelerate the algorithm [10]. Chen et al. [8] propose a community-based alignment methods that can not only leverage the node attributes, but also find both the node-level alignment and the community-level alignment. However, most, if not all, of the existing methods implicitly assume that the input networks are complete without missing edges.

**Network Completion.** On the other side, since the real-world networks are often incomplete, the network completion task is often the very first step prior to many applications in order to gain a better performance. Kim et al. propose a network completion method based on expectation maximization (EM) that can add the missing nodes and edges under the assumption that networks follow the Kronecker graph model [17]. Another work proposed by Masrour et al. decouples the network completion from transduction so that the node similarity matrix can be efficiently leveraged as the side information [27]. Soundarajan et al. study to reduce the incompleteness of the input network by a careful node selection to probe nodes [35]. In addition, inferring the missing edges in the incomplete network can be considered as an adjacency matrix completion problem, and hence the network completion problem can be naturally solved by many matrix completion approaches. Among them, one well-known method is based on singular value thresholding (SVT) which iteratively shrinkages the singular values to minimize the nuclear norm [7]. In order to speed up the completion process, Toh and Yun propose an accelerated proximal gradient algorithm to solve the nuclear norm regularized linear least squares problem [36]. Besides, by exploiting the geometry of the low-rank structure constraint, a first-order and second-order Riemannian trust-region approach is proposed to solve the formulated optimization problem on the Grassmann manifold [5]. [37] utilizes graph auto-encoder for matrix completion, or specifically for bipartite network completion. However, all these methods aim to complete a single network once at a time. On the other hand, there are some recent work to complete multiple *aligned* networks by tensor completion [22, 24]. Nonetheless, how these input networks are aligned at the first place was not answered in these work.

Another hot topic related to network completion is the link prediction problem [21]. Miller et al. propose a Beysian nonparametric latent feature model to predict the links in the relational data [29]. [28] proposes to use the supervised matrix factorization to obtain the latent features and then predict links by combining latent features with the explicit node and edge features. Barbieri et al. propose a stochastic topic model which can not only predict the links to be recommended, but also explain why the links are predicted [2]. In [41], Zhang et al. study the link prediction problem across multiple aligned networks and formulate it as a sparse low-rank matrix completion problem. This is different from INEAT where no alignment are known apriori.

# 7 CONCLUSION

In the era of big data, the multi-sourced and incomplete characteristics often *co-exist* in many real networks. Nonetheless, the state-of-the-arts has been largely addressing them *in parallel*. In this paper, we propose to jointly address network alignment and network completion so that the two tasks can mutually benefit from each other. We formulate incomplete network alignment problem as an optimization problem and propose a multiplicative update algorithm (INEAT) to solve it. The proposed algorithm is proved to converge to the KKT fixed point with a linear complexity in both time and space. To our best knowledge, the proposed iNEAT algorithm is the first network alignment algorithm with a provable linear complexity. The empirical evaluations demonstrate the effectiveness and efficiency of the proposed iNEAT algorithm. Specially, it (1) improves the alignment accuracy by up to 30% over the existing network alignment methods, in the meanwhile leads a better imputation outcome; and (2) achieves a good quality-speed balance and scales *linearly* w.r.t the number of nodes in the networks. Future work includes extending our algorithm to handle attributed networks and other ways to leverage the prior knowledge.

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<sup>&</sup>lt;sup>4</sup>Distribution Statement "A" (Approved for Public Release, Distribution Unlimited)

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## A UPDATE RULE OF $V_1, U_2, V_2$

Similarly, the gradient of Eq. (2) in terms of  $V_1$  is computed by  $\frac{\partial J_1}{\partial V_1} = X_6 - Y_6$  where

$$\mathbf{X}_{6} = [\mathbf{P}_{\Omega_{1}} \odot (\mathbf{V}_{1}\mathbf{U}_{1}^{T})]\mathbf{U}_{1} + \lambda \mathbf{V}_{1}$$
$$\mathbf{Y}_{6} = (\mathbf{P}_{\Omega_{1}} \odot \mathbf{A}_{1})\mathbf{U}_{1}$$

The partial gradient of Eq. (9) w.r.t.  $V_1$  is computed by  $\frac{\partial J_2}{\partial V_1} = X_7 - Y_7$  where

$$\mathbf{X}_{7} = \frac{\gamma}{2} \mathbf{1}_{1} \mathbf{1}_{r_{1}}^{T} [(\mathbf{M}^{T} \mathbf{U}_{2}^{T} \mathbf{D}_{2} \mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T}) \odot \mathbf{U}_{1}^{T}] \mathbf{U}_{1} + \frac{\gamma}{2} [(\mathbf{U}_{1} \mathbf{M}^{T} \mathbf{U}_{2}^{T} \hat{\mathbf{D}}_{2} \mathbf{U}_{2} \mathbf{M}) \odot \mathbf{U}_{1}] \mathbf{1}_{r_{1}} \mathbf{1}_{1}^{T} \mathbf{U}_{1}$$
  
$$\mathbf{Y}_{7} = \gamma \mathbf{U}_{1} \mathbf{M}^{T} \mathbf{U}_{2}^{T} \mathbf{V}_{2} \mathbf{U}_{2}^{T} \mathbf{U}_{2} \mathbf{M}$$

And  $\frac{\partial J_3}{\partial V_1}$  is computed by  $\frac{\partial J_3}{\partial V_1} = X_8 - Y_8$  where

$$\begin{aligned} \mathbf{X}_8 &= \beta [\mathbf{P}_{\bar{\Omega}_1} \odot (\mathbf{V}_1 \mathbf{U}_1^T)] \mathbf{U}_1 \\ \mathbf{Y}_8 &= \beta [\mathbf{P}_{\bar{\Omega}_1} \odot (\mathbf{U}_1 \mathbf{M}^T \mathbf{U}_2^T \mathbf{A}_2 \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T)] \mathbf{U}_1 \end{aligned}$$

The fixed-point solution of  $\frac{\partial J}{\mathbf{V}_1} = \mathbf{0}$  under the nonnegativity constraint of  $\mathbf{V}_1$  leads to the following multiplicative update rule

$$\mathbf{V}_{1}(p,q) \leftarrow \mathbf{V}_{1}(p,q) \sqrt[4]{\frac{\mathbf{Y}_{6}(p,q) + \mathbf{Y}_{7}(p,q) + \mathbf{Y}_{8}(p,q)}{\mathbf{X}_{6}(p,q) + \mathbf{X}_{7}(p,q) + \mathbf{X}_{8}(p,q)}}$$
(27)

The gradient of Eq. (2) w.r.t.  $U_2$  is computed by  $\frac{\partial f_1}{\partial U_2} = X_9 - Y_9$  where

$$\mathbf{X}_9 = [\mathbf{P}_{\Omega_1} \odot (\mathbf{U}_2 \mathbf{V}_2^T)] \mathbf{V}_2 + \lambda \mathbf{U}_2$$
$$\mathbf{Y}_9 = (\mathbf{P}_{\Omega_1} \odot \mathbf{A}_2) \mathbf{V}_2$$

For Eq. (9), its gradient over U<sub>2</sub> can be derived as  $\frac{\partial J_2}{\partial U_2} = \mathbf{X}_{10} - \mathbf{Y}_{10}$  where

$$\begin{split} \mathbf{X}_{10} &= \frac{\gamma}{2} \big[ \mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T} \mathbf{D}_{1} \mathbf{U}_{1} \mathbf{M}^{T} \big) \odot \mathbf{U}_{2} \big] \mathbf{1}_{r_{2}} \mathbf{1}_{2}^{T} \mathbf{V}_{2} + \frac{\gamma}{2} \mathbf{1}_{2} \mathbf{1}_{r_{2}}^{T} \big[ (\mathbf{M} \mathbf{U}_{1}^{T} \hat{\mathbf{D}}_{1} \mathbf{U}_{1} \mathbf{M}^{T} \mathbf{U}_{2}^{T} \big) \odot \mathbf{U}_{2}^{T} \big] \\ &+ \gamma (\gamma \mathbf{D}_{2} \mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T} \mathbf{D}_{1} \mathbf{U}_{1} \mathbf{M}^{T} + \hat{\mathbf{D}}_{2} \mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T} \hat{\mathbf{D}}_{1} \mathbf{U}_{1} \mathbf{M}^{T} \big) \\ \mathbf{Y}_{10} &= \gamma \mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T} \mathbf{U}_{1} \mathbf{V}_{1}^{T} \mathbf{U}_{1} \mathbf{M}^{T} \mathbf{U}_{2}^{T} \mathbf{V}_{2} + \gamma \mathbf{V}_{2} \mathbf{U}_{2}^{T} \mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T} \mathbf{U}_{1} \mathbf{M}^{T} + \gamma \mathbf{U}_{2} \mathbf{V}_{2}^{T} \mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T} \mathbf{U}_{1} \mathbf{M}^{T} \end{split}$$

The partial gradient of Eq. (11) w.r.t. U<sub>2</sub> is computed by  $\frac{\partial f_3}{\partial U_2} = X_{11} - Y_{11}$  where

$$\begin{split} \mathbf{X}_{11} &= 2\beta [\mathbf{P}_{\bar{\Omega}_2} \odot (\mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \mathbf{A}_1 \mathbf{U}_1 \mathbf{M}^T \mathbf{U}_2^T)] \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \mathbf{A}_1 \mathbf{U}_1 \mathbf{M}^T + \beta [\mathbf{P}_{\bar{\Omega}_2} \odot (\mathbf{U}_2 \mathbf{V}_2^T)] \mathbf{V}_2 \\ &+ 2\beta \mathbf{A}_2 \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T [\mathbf{P}_{\bar{\Omega}_1} \odot (\mathbf{U}_1 \mathbf{M}^T \mathbf{U}_2^T \mathbf{A}_2 \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T)] \mathbf{U}_1 \mathbf{M}^T \\ \mathbf{Y}_{11} &= \beta [\mathbf{P}_{\bar{\Omega}_2} \odot (\mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \mathbf{A}_1 \mathbf{U}_1 \mathbf{M}^T \mathbf{U}_2^T)] \mathbf{V}_2 + \beta [\mathbf{P}_{\bar{\Omega}_2} \odot (\mathbf{U}_2 \mathbf{V}_2^T + \mathbf{V}_2 \mathbf{U}_2^T)] \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \mathbf{A}_1 \mathbf{U}_1 \mathbf{M}^T \\ &+ \beta \mathbf{A}_2 \mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T [\mathbf{P}_{\bar{\Omega}_1} \odot (\mathbf{U}_1 \mathbf{V}_1^T + \mathbf{V}_1 \mathbf{U}_1^T)] \mathbf{U}_1 \mathbf{M}^T \end{split}$$

In this way, the fixed-point solution of  $\frac{\partial J}{\partial U_2} = \mathbf{0}$  under the nonnegativity constraint of  $U_2$  gives the following multiplicative update rule

$$\mathbf{U}_{2}(p,q) \leftarrow \mathbf{U}_{2}(p,q) \sqrt[4]{\frac{\mathbf{Y}_{9}(p,q) + \mathbf{Y}_{10}(p,q) + \mathbf{Y}_{11}(p,q)}{\mathbf{X}_{9}(p,q) + \mathbf{X}_{10}(p,q) + \mathbf{X}_{11}(p,q)}}$$
(28)

The gradient of Eq. (2) w.r.t.  $\mathbf{V}_2$  is computed by  $\frac{\partial f_1}{\partial \mathbf{V}_2} = \mathbf{X}_{12} - \mathbf{Y}_{12}$  where

$$\mathbf{X}_{12} = [\mathbf{P}_{\Omega_2} \odot (\mathbf{U}_2 \mathbf{V}_2^T)]\mathbf{U}_2 + \lambda \mathbf{V}_2$$
$$\mathbf{Y}_{12} = (\mathbf{P}_{\Omega_2} \odot \mathbf{A}_2)\mathbf{U}_2$$

The gradient of Eq. (9) over  $\mathbf{V}_2$  is computed by  $\frac{\partial f_2}{\partial \mathbf{V}_2} = \mathbf{X}_{13} - \mathbf{Y}_{13}$  where

$$\begin{aligned} \mathbf{X}_{13} &= \frac{\gamma}{2} \mathbf{1}_{2} \mathbf{1}_{r_{2}}^{T} [(\mathbf{M} \mathbf{U}_{1}^{T} \mathbf{D}_{1} \mathbf{U}_{1} \mathbf{M}^{T} \mathbf{U}_{2}^{T}) \odot \mathbf{U}_{2}^{T}] \mathbf{U}_{2} + \frac{\gamma}{2} [(\mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T} \hat{\mathbf{D}}_{1} \mathbf{U}_{1} \mathbf{M}^{T}) \odot \mathbf{U}_{2}] \mathbf{1}_{r_{2}} \mathbf{1}_{2}^{T} \mathbf{U}_{2} \\ \mathbf{Y}_{13} &= \gamma \mathbf{U}_{2} \mathbf{M} \mathbf{U}_{1}^{T} \mathbf{V}_{1} \mathbf{U}_{1}^{T} \mathbf{U}_{1} \mathbf{M}^{T} \mathbf{U}_{2}^{T} \mathbf{U}_{2} \end{aligned}$$

And the gradient of Eq. (11) w.r.t.  $\mathbf{V}_2$  is  $\frac{\partial f_3}{\partial \mathbf{V}_2} = \mathbf{X}_{14} - \mathbf{Y}_{14}$  where

$$\begin{aligned} \mathbf{X}_{14} &= \beta [\mathbf{P}_{\bar{\Omega}_2} \odot (\mathbf{V}_2 \mathbf{U}_2^T)] \mathbf{U}_2 \\ \mathbf{Y}_{14} &= \beta [\mathbf{P}_{\bar{\Omega}_2} \odot (\mathbf{U}_2 \mathbf{M} \mathbf{U}_1^T \mathbf{A}_1 \mathbf{U}_1 \mathbf{M}^T \mathbf{U}_2^T)] \mathbf{U}_2 \end{aligned}$$

In this way, the fixed-point solution of  $\frac{\partial J}{\partial V_2} = 0$  under the nonnegativity constraint of  $V_2$  leads to the multiplicative update rule as

$$\mathbf{V}_{2}(p,q) \leftarrow \mathbf{V}_{2}(p,q) \sqrt[4]{\frac{\mathbf{Y}_{12}(p,q) + \mathbf{Y}_{13}(p,q) + \mathbf{Y}_{14}(p,q)}{\mathbf{X}_{12}(p,q) + \mathbf{X}_{13}(p,q) + \mathbf{X}_{14}(p,q)}}$$
(29)

#### B PROOF OF LEMMA 2

PROOF. First, we prove that for any nonnegative  $U_1, \tilde{U}_1$ , we have  $Z(U_1, \tilde{U}_1) \ge J(U_1)$ . Recall that the objective function w.r.t.  $U_1$  contains three parts, i.e.,  $J(U_1) = J_1(U_1) + J_2(U_1) + J_3(U_1)$ . For  $J_1(U_1)$ , after removing the constant terms w.r.t.  $U_1$ , we have

$$J_1(\mathbf{U}_1) = \underbrace{\frac{1}{2} \|\mathbf{P}_{\Omega_1} \odot (\mathbf{U}_1 \mathbf{V}_1^T)\|_F^2}_{T_1} \underbrace{-\operatorname{Tr}(\mathbf{A}_1 [\mathbf{P}_{\Omega_1} \odot (\mathbf{V}_1 \mathbf{U}_1^T)])}_{T_2} + \underbrace{\frac{\lambda}{2} \|\mathbf{U}_1\|_F^2}_{T_3}$$

# Similarly, we have the following for $J_2(U_1)$ and $J_3(U_1)$

$$J_{2}(\mathbf{U}_{1}) = \underbrace{\underbrace{\frac{\gamma}{2}}_{2} \operatorname{Tr}(\mathbf{U}_{1}^{T}\mathbf{D}_{1}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{D}_{2}\mathbf{U}_{2}\mathbf{M} + \mathbf{U}_{1}^{T}\hat{\mathbf{D}}_{1}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\hat{\mathbf{D}}_{2}\mathbf{U}_{2}\mathbf{M})}_{T_{4}} \\ \underbrace{-\gamma \operatorname{Tr}(\mathbf{U}_{2}\mathbf{V}_{2}^{T}\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}\mathbf{V}_{1}\mathbf{U}_{1}^{T}\mathbf{V}_{1}\mathbf{U}_{1}^{T}\mathbf{U}_{1}\mathbf{M}\mathbf{U}_{2}^{T})}_{T_{5}} \\ J_{3}(\mathbf{U}_{1}) = \underbrace{\frac{\beta}{2}}_{2} \|\mathbf{P}_{\bar{\Omega}_{1}} \odot (\mathbf{U}_{1}\mathbf{V}_{1}^{T})\|_{F}^{2} + \underbrace{\frac{\beta}{2}}_{2} \|\mathbf{P}_{\bar{\Omega}_{1}} \odot (\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T})\|_{F}^{2}}_{T_{6}} + \underbrace{\frac{\beta}{2}}_{T_{8}} \|\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}\mathbf{A}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T})\|_{F}^{2}}_{T_{8}} \\ \underbrace{-\beta \operatorname{Tr}([\mathbf{P}_{\bar{\Omega}_{1}} \odot (\mathbf{U}_{1}\mathbf{V}_{1}^{T})][\mathbf{P}_{\bar{\Omega}_{1}} \odot (\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T})])}_{T_{9}}_{T_{10}} \\ \underbrace{-\beta \operatorname{Tr}([\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{V}_{2}^{T})][\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}\mathbf{A}_{1}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T})])}_{T_{10}} \\ \underbrace{-\beta \operatorname{Tr}([\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{V}_{2}^{T})][\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}\mathbf{A}_{1}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T})])}_{T_{10}} \\ \underbrace{-\beta \operatorname{Tr}([\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{V}_{2}^{T})][\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}\mathbf{A}_{1}\mathbf{U}_{1}\mathbf{M}^{T}\mathbf{U}_{2}^{T})])}_{T_{10}} \\ \underbrace{-\beta \operatorname{Tr}([\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{V}_{2}^{T})][\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T})])}_{T_{10}} \\ \underbrace{-\beta \operatorname{Tr}([\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{V}_{2}^{T})][\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}\mathbf{A}_{2}\mathbf{U}_{2}\mathbf{M}\mathbf{U}_{1}^{T}\mathbf{U}_{2}^{T})]])}_{T_{10}} \\ \underbrace{-\beta \operatorname{Tr}([\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf{V}_{2}^{T})][\mathbf{P}_{\bar{\Omega}_{2}} \odot (\mathbf{U}_{2}\mathbf$$

Now we prove that  $T'_i \ge T_i$  for  $i = 1, \dots, 10$  term by term. By the definition of  $\mathbf{P}_{\Omega_1}$  and its symmetry,  $\mathbf{P}_{\Omega_1} = \mathbf{P}_{\Omega_1}^T \ge \mathbf{P}_{\Omega_1} \odot \mathbf{P}_{\Omega_1}$ . Let  $\mathbf{U}_1(p,q) = \tilde{\mathbf{U}}_1(p,q)\mathbf{Q}(p,q), \forall p,q$ , we have

$$\begin{split} T_{1}' &\geq \frac{1}{2} \sum_{p,q} \left[ (\mathbf{P}_{\Omega_{1}} \odot \mathbf{P}_{\Omega_{1}} (\tilde{\mathbf{U}}_{1} \mathbf{V}_{1}^{T})) \mathbf{V}_{1} \right] (p,q) \frac{\mathbf{U}_{1}^{2}(p,q)}{\tilde{\mathbf{U}}_{1}(p,q)} \\ &= \frac{1}{2} \sum_{p,q,r,s} \mathbf{P}_{\Omega_{1}}^{2}(p,r) \mathbf{V}_{1}(r,t) \mathbf{V}_{1}(r,q) \tilde{\mathbf{U}}_{1}(p,t) \tilde{\mathbf{U}}_{1}(p,q) \mathbf{Q}^{2}(p,q) \\ &= \frac{1}{2} \sum_{p,q,r,s} \mathbf{P}_{\Omega_{1}}^{2}(p,r) \mathbf{V}_{1}(r,t) \mathbf{V}_{1}(r,q) \tilde{\mathbf{U}}_{1}(p,t) \tilde{\mathbf{U}}_{1}(p,q) (\frac{\mathbf{Q}^{2}(p,t) + \mathbf{Q}^{2}(p,q)}{2}) \\ &\geq \frac{1}{2} \sum_{p,q,r,s} \mathbf{P}_{\Omega_{1}}^{2}(p,r) \mathbf{V}_{1}(r,t) \mathbf{V}_{1}(r,q) \tilde{\mathbf{U}}_{1}(p,t) \tilde{\mathbf{U}}_{1}(p,q) \mathbf{Q}(p,t) \mathbf{Q}(p,q) \\ &= \frac{1}{2} \sum_{p,q,r,s} \mathbf{P}_{\Omega_{1}}^{2}(p,r) \mathbf{V}_{1}(r,t) \mathbf{V}_{1}(r,q) \mathbf{U}_{1}(p,t) \mathbf{U}_{1}(p,q) = T_{1} \end{split}$$

where the first and fourth line is due to  $a^2 + b^2 \ge 2ab$  and the third line is due to the equivalence by switching  $t \Leftrightarrow q$ . We can prove  $T'_6 \ge T_6$  in the same way.

Next, by using the inequality  $z \ge 1 + \log z$ ,  $\forall z > 0$ , we can easily prove  $T'_2 \ge T_2$ . And similarly,  $T'_5 \ge T_5, T'_9 \ge T_9$  and  $T'_{10} \ge T_{10}$ .

For  $T'_3$ , we have  $T'_3 \ge \frac{\lambda}{2} \sum_{p,q} \mathbf{U}_1^2(p,q) = T_3$ .

To show  $T'_6 \ge T_6$ , we first consider the first term of  $T_6$ . By denoting  $\mathbf{X} = \mathbf{M}^T \mathbf{U}_2^T \mathbf{D}_2 \mathbf{U}_2 \mathbf{M}$  which is symmetric, we have

$$\frac{\gamma}{2} \operatorname{Tr}(\mathbf{D}_{1}\mathbf{U}_{1}\mathbf{X}\mathbf{U}_{1}^{T}) = \frac{\gamma}{2} \sum_{p,q,r,s} \mathbf{U}_{1}(p,r)(\mathbf{V}_{1}^{T}\mathbf{1}_{1})(r,1)\mathbf{U}_{1}(p,s)\mathbf{X}(s,q)\mathbf{U}_{1}(p,q) \\
= \frac{\gamma}{2} \sum_{p,q,r,s} \tilde{\mathbf{U}}_{1}(p,r)(\mathbf{V}_{1}^{T}\mathbf{1}_{1})(r,1)\tilde{\mathbf{U}}_{1}(p,s)\mathbf{X}(s,q)\tilde{\mathbf{U}}_{1}(p,q)\mathbf{Q}(p,r)\mathbf{Q}(p,s)\mathbf{Q}(p,q) \\
\leq \frac{\gamma}{2} \sum_{p,q,r,s} \tilde{\mathbf{U}}_{1}(p,r)(\mathbf{V}_{1}^{T}\mathbf{1}_{1})(r,1)\tilde{\mathbf{U}}_{1}(p,s)\mathbf{X}(s,q)\tilde{\mathbf{U}}_{1}(p,q)(\frac{\mathbf{Q}(p,r)^{3} + \mathbf{Q}(p,s)^{3} + \mathbf{Q}(p,q)^{3}}{3}) \\
= \frac{\gamma}{6} \sum_{p,r} [((\tilde{\mathbf{U}}_{1}\mathbf{X}) \odot \tilde{\mathbf{U}}_{1})\mathbf{1}_{r_{1}}\mathbf{1}_{1}^{T}\mathbf{V}_{1}](p,r)\frac{\mathbf{U}_{1}^{3}(p,r)\tilde{\mathbf{U}}_{1}(p,r)}{\tilde{\mathbf{U}}_{1}^{3}(p,r)} + \frac{\gamma}{3} \sum_{p,q} [\operatorname{diag}(\tilde{\mathbf{U}}_{1}\mathbf{V}_{1}^{T}\mathbf{1})\tilde{\mathbf{U}}_{1}\mathbf{X}](p,q)\frac{\mathbf{U}_{1}^{3}(p,q)\tilde{\mathbf{U}}_{1}(p,q)}{\tilde{\mathbf{U}}_{1}^{3}(p,q)} \\
\leq \frac{\gamma}{12} \sum_{p,q} [\frac{1}{2}((\tilde{\mathbf{U}}_{1}\mathbf{X}) \odot \tilde{\mathbf{U}}_{1})\mathbf{1}_{r_{1}}\mathbf{1}_{1}^{T}\mathbf{V}_{1} + \operatorname{diag}(\tilde{\mathbf{U}}_{1}\mathbf{V}_{1}^{T}\mathbf{1})\tilde{\mathbf{U}}_{1}\mathbf{X}](p,q)\frac{3\mathbf{U}_{1}^{4}(p,q) + \tilde{\mathbf{U}}_{1}^{4}(p,q)}{\tilde{\mathbf{U}}_{1}^{3}(p,q)} \\ \leq \frac{\gamma}{12} \sum_{p,q} [\frac{1}{2}((\tilde{\mathbf{U}}_{1}\mathbf{X}) \odot \tilde{\mathbf{U}}_{1})\mathbf{1}_{r_{1}}\mathbf{1}_{1}^{T}\mathbf{V}_{1} + \operatorname{diag}(\tilde{\mathbf{U}}_{1}\mathbf{V}_{1}^{T}\mathbf{1})\tilde{\mathbf{U}}_{1}\mathbf{X}](p,q)\frac{3\mathbf{U}_{1}^{4}(p,q) + \tilde{\mathbf{U}}_{1}^{4}(p,q)}{\tilde{\mathbf{U}}_{1}^{3}(p,q)} \\ \leq \frac{\gamma}{12} \sum_{p,q} [\frac{1}{2}((\tilde{\mathbf{U}}_{1}\mathbf{X}) \odot \tilde{\mathbf{U}}_{1})\mathbf{1}_{r_{1}}\mathbf{1}_{1}^{T}\mathbf{V}_{1} + \operatorname{diag}(\tilde{\mathbf{U}}_{1}\mathbf{V}_{1}^{T}\mathbf{1})\tilde{\mathbf{U}}_{1}\mathbf{X}](p,q)\frac{3\mathbf{U}_{1}^{4}(p,q) + \tilde{\mathbf{U}}_{1}^{4}(p,q)}{\tilde{\mathbf{U}}_{1}^{3}(p,q)} \\ \leq \frac{\gamma}{12} \sum_{p,q} [\frac{1}{2}((\tilde{\mathbf{U}}_{1}\mathbf{X}) \odot \tilde{\mathbf{U}}_{1})\mathbf{1}_{r_{1}}\mathbf{U}_{1}^{T}\mathbf{U}_{1} + \operatorname{diag}(\tilde{\mathbf{U}}_{1}\mathbf{V}_{1}^{T}\mathbf{U}_{1})\mathbf{U}_{1}\mathbf{U}_{1}) \\ \leq \frac{\gamma}{12} \sum_{p,q} [\frac{1}{2}((\tilde{\mathbf{U}}_{1}\mathbf{X}) \odot \tilde{\mathbf{U}}_{1})\mathbf{U}_{1}\mathbf{U}_{1}] + \operatorname{diag}(\tilde{\mathbf{U}}_{1}\mathbf{U}_{1}^{T}\mathbf{U}_{1})\mathbf{U}_{1}\mathbf{U}_{1}) \\ \leq \frac{\gamma}{12} \sum_{p,q} [\frac{1}{2}((\tilde{\mathbf{U}}_{1}\mathbf{X}) \odot \tilde{\mathbf{U}}_{1})\mathbf{U}_{1}\mathbf{U}_{1}] + \operatorname{diag}(\tilde{\mathbf{U}}_{1}\mathbf{U}_{1})\mathbf{U}_{1}\mathbf{U}_{1})\mathbf{U}_{1}\mathbf{U}_{1}) \\ \leq \frac{\gamma}{12} \sum_{p,q} [\frac{1}{2}((\tilde{\mathbf{U}}_{1}\mathbf{X}) \odot \tilde{\mathbf{U}}_{1})\mathbf{U}_{1}\mathbf{U}_{1}] \\ \leq \frac{\gamma}{12} \sum_{p,q} [\frac{1}{2}(\tilde{\mathbf{U}}_{1}\mathbf{U}_{1})\mathbf{U}_{1}\mathbf{U}_{1}] \\ \leq \frac{\gamma}{12} \sum_{p,q} [\frac{1}{2}(\tilde{\mathbf{U}}_{1}\mathbf{U}_{1})\mathbf{U}_{1}\mathbf{U}$$

Similarly, by denoting  $\hat{\mathbf{X}} = \mathbf{M}^T \mathbf{U}_2^T \hat{\mathbf{D}}_2 \mathbf{U}_2 \mathbf{M}$ , we can show that

$$\frac{\gamma}{2} \operatorname{Tr}(\hat{\mathbf{D}}_{1}\mathbf{U}_{1}\hat{\mathbf{X}}\mathbf{U}_{1}^{T}) \leq \frac{\gamma}{12} \sum_{p,q} \left[\frac{1}{2}\mathbf{1}_{1}\mathbf{1}_{r_{1}}^{T}(\tilde{\mathbf{U}}_{1}^{T} \odot(\hat{\mathbf{X}}\tilde{\mathbf{U}}_{1}^{T}))\mathbf{V}_{1} + \operatorname{diag}(\mathbf{V}_{1}\tilde{\mathbf{U}}_{1}^{T}\mathbf{1}_{1})\tilde{\mathbf{U}}_{1}\hat{\mathbf{X}}\right](p,q) \frac{3\mathbf{U}_{1}^{4}(p,q) + \mathbf{U}_{1}^{4}(p,q)}{\tilde{\mathbf{U}}_{1}^{3}(p,q)}$$

$$(31)$$

Then by adding Eq. (30) and Eq. (31) together, we can obtain  $T_6 \leq T'_6$ . Denote  $\mathbf{T} = \mathbf{M}^T \mathbf{U}_2^T \mathbf{A}_2 \mathbf{U}_2 \mathbf{M}$ , we have

$$\begin{split} T_7' &= \frac{\beta}{2} \sum_{p,q} \left[ (\mathbf{P}_{\bar{\Omega}_1} \odot (\tilde{\mathbf{U}}_1 \mathbf{T} \tilde{\mathbf{U}}_1^T)) \tilde{\mathbf{U}}_1 \mathbf{T} \right](p,q) \frac{\mathbf{U}_1^4(p,q)}{\tilde{\mathbf{U}}_1^3(p,q)} \\ &= \frac{\beta}{2} \sum_{p,q,r,s,t,u} \mathbf{P}_{\bar{\Omega}_1}(p,r) \mathbf{T}(s,t) \mathbf{T}(u,q) \tilde{\mathbf{U}}_1(p,s) \tilde{\mathbf{U}}_1(r,t) \tilde{\mathbf{U}}_1(r,u) \tilde{\mathbf{U}}_(p,q) \mathbf{Q}^4(p,q) \end{split}$$

Due to the symmetry of  $\mathbf{P}_{\bar{\Omega}_1}$  and the fact that  $\mathbf{P}_{\bar{\Omega}_1} = \mathbf{P}_{\bar{\Omega}_1} \odot \mathbf{P}_{\bar{\Omega}_1}$ , by switching the indices  $s \Leftrightarrow t, u \Leftrightarrow$  $q, r \Leftrightarrow p$ , we have

$$T_7' = \frac{\beta}{2} \sum_{p,q,r,s,t,u} \mathbf{P}_{\bar{\Omega}_1}^2(p,r) \mathbf{T}(s,t) \mathbf{T}(u,q) \tilde{\mathbf{U}}_1(p,s) \tilde{\mathbf{U}}_1(r,t) \times \tilde{\mathbf{U}}_1(r,u) \tilde{\mathbf{U}}_1(p,q) \mathbf{Q}^4(r,u)$$

And we can similar results by switching  $s \Leftrightarrow u, t \Leftrightarrow q, p \Leftrightarrow r$  and  $s \Leftrightarrow q, t \Leftrightarrow u, p \Leftrightarrow r$ . In this way, we have

$$\begin{split} T_{7}' &= \frac{\beta}{2} \sum_{p,q,r,s,t,u} \mathbf{P}_{\bar{\Omega}_{1}}^{2}(p,r)\mathbf{T}(s,t)\mathbf{T}(u,q)\tilde{\mathbf{U}}_{1}(p,s)\tilde{\mathbf{U}}_{1}(r,t)\tilde{\mathbf{U}}_{1}(r,u)\tilde{\mathbf{U}}_{1}(p,q) \\ &\times [\frac{\mathbf{Q}^{4}(p,q) + \mathbf{Q}^{4}(r,u) + \mathbf{Q}^{4}(r,t) + \mathbf{Q}^{4}(p,s)}{4}] \\ &\geq \frac{\beta}{2} \sum_{p,q,r,s,t,u} \mathbf{P}_{\bar{\Omega}_{1}}^{2}(p,r)\mathbf{T}(s,t)\mathbf{T}(u,q)\tilde{\mathbf{U}}_{1}(p,s)\tilde{\mathbf{U}}_{1}(r,t)\tilde{\mathbf{U}}_{1}(r,u)\tilde{\mathbf{U}}_{1}(p,q)\mathbf{Q}(p,q)\mathbf{Q}(r,u)\mathbf{Q}(r,t)\mathbf{Q}(p,s) \\ &= \frac{\beta}{2} \sum_{p,q,r,s,t,u} \mathbf{P}_{\bar{\Omega}_{1}}^{2}(p,r)\mathbf{T}(s,t)\mathbf{T}(u,q)\mathbf{U}_{1}(p,s)\mathbf{U}_{1}(r,t)\mathbf{U}_{1}(r,u)\mathbf{U}_{1}(p,s) \\ &= T_{7} \end{split}$$

Similarly, it can be proved that  $T'_8 \ge T_8$ . Then by adding  $T'_i$ ,  $i = 1, \dots, 10$  together, we have

$$Z(\mathbf{U}_1, \tilde{\mathbf{U}}_1) = \sum_{i=1}^{10} T_i' \ge \sum_{i=1}^{10} T_i = J(\mathbf{U}_1)$$
(32)

Second, we can directly prove  $Z(U_1, U_1) = J(U_1)$  by substituting  $\tilde{U}_1$  with  $U_1$  in all above inequalities.

Finally, to prove that the auxiliary function  $Z(\mathbf{U}_1, \tilde{\mathbf{U}}_1)$  is convex w.r.t.  $\mathbf{U}_1$ , for the sake of brevity, we briefly show the Hessian matrix of  $Z(\mathbf{U}_1, \tilde{\mathbf{U}}_1)$  is positive semi-definite. To start with, the derivative  $\frac{\partial Z(\mathbf{U}_1, \tilde{\mathbf{U}}_1)}{\partial \mathbf{U}_1(p,q)}$  can be calculated as

$$\frac{\partial Z(\mathbf{U}_1, \tilde{\mathbf{U}}_1)}{\partial \mathbf{U}_1(p, q)} = (\tilde{\mathbf{X}}_1 + \tilde{\mathbf{X}}_2 + \tilde{\mathbf{X}}_3)(p, q) \frac{\mathbf{U}_1^3(p, q)}{\tilde{\mathbf{U}}_1^3(p, q)} - (\tilde{\mathbf{Y}}_1 + \tilde{\mathbf{Y}}_2 + \tilde{\mathbf{Y}}_3)(p, q) \frac{\tilde{\mathbf{U}}_1(p, q)}{\mathbf{U}_1(p, q)}$$
(33)

where  $\tilde{\mathbf{X}}_i$ ,  $\tilde{\mathbf{Y}}_i$ , i = 1, 2, 3 are all in terms of  $\tilde{\mathbf{U}}_1$  while sharing the same formulas with  $\mathbf{X}_i$ ,  $\mathbf{Y}_i$ , i = 1, 2, 3. For example,  $\tilde{\mathbf{X}}_1 = [\mathbf{P}_{\Omega_1} \odot (\tilde{\mathbf{U}}_1 \mathbf{V}_1^T)] \mathbf{V}_1 + \lambda \tilde{\mathbf{U}}_1$ . Then the Hessian matrix w.r.t.  $\mathbf{U}_1$  is computed by

$$\frac{\partial^2 Z(\mathbf{U}_1, \tilde{\mathbf{U}}_1)}{\partial U_1(p, q) \partial U_1(r, s)} = \delta_{pr} \delta_{qs} \left[ 3(\tilde{\mathbf{X}}_1 + \tilde{\mathbf{X}}_2 + \tilde{\mathbf{X}}_3)(p, q) \frac{\mathbf{U}_1^2(p, q)}{\tilde{\mathbf{U}}_1^3(p, q)} + (\tilde{\mathbf{Y}}_1 + \tilde{\mathbf{Y}}_2 + \tilde{\mathbf{Y}}_3)(p, q) \frac{\tilde{\mathbf{U}}_1(p, q)}{\mathbf{U}_1^2(p, q)} \right]$$
(34)

where  $\delta_{pr}$ ,  $\delta_{qs}$  are the Kronecker delta functions, i.e.,  $\delta_{pr} = 1$  if p = r;  $\delta_{pr} = 0$  otherwise. Therefore, the Hessian matrix  $\nabla^2_{\mathbf{U}_1} Z(\mathbf{U}_1, \tilde{\mathbf{U}}_1)$  is a diagonal matrix with nonnegative elements. As a result, the Hessian matrix is positive semi-definite, which means the auxiliary function  $Z(\mathbf{U}_1, \tilde{\mathbf{U}}_1)$  is convex w.r.t.  $\mathbf{U}_1$ . In this way, the global minima of  $Z(\mathbf{U}_1, \tilde{\mathbf{U}}_1)$  is obtained by setting its first-order derivative (i.e., Eq. (33)) to be zero which further leads to the solution consistent with Eq. (26).

39:26